SUPERCONNECTIONS AND PARALLEL TRANSPORT

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Abstract

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We construct a notion of parallel transport along superpaths in a manifold that corresponds to a superconnection (à la Quillen), in an attempt to understand geometrically superconnections, the same way as an appropriate notion of parallel transport along paths translates geometrically the concept of a connection. The parallel transport along superpaths is realized by solving some "half-order" differential equations, as opposed to solving first-order differential equations for the usual parallel transport. Before doing this, we extend the usual notion of parallel transport along paths associated to a connection to superpaths, and see how the super-parallel transport incorporates the analytical concept of a connection. Such considerations are motivated by trying to understand one dimensional supersymmetric field theories over a manifold, in the hope that they provide geometric cocycles for differential K-theory. The larger context is the Stolz-Teichner program (see [ST04]) of relating field theories and cohomology theories, and our effort is to complete the understanding of the one-dimensional story.

To my teachers.

Alor mei.

CONTENTS

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CHAPTER 1

INTRODUCTION

A connection on a vector bundle over a manifold M can be understood *geometrically* as allowing to identify between various fibers of the bundle along paths in the manifold M. Such a construction is called in general *parallel transport* based on M. This geometric picture can then be interpreted as giving rise to a (topological) 1-dimensional field theory based on M. More generally, a connection together with a bundle endomorphism can also be understood as a parallel transport. This gives rise to a (euclidean) 1-dim field theory over M. “Topological” here means that the identification between various fibers along paths does *not* depend on the parametrization of the interval, or, better yet said, the identification does not depend on the metric structure on the parametrizing intervals, whereas “euclidean” means that it does.

In this paper we interpret superconnections geometrically. They will give rise to a notion of parallel transport along superpaths in M. Recall (see [BGV92], Chapter 1) that a superconnection (aka Quillen superconnection) on a vector bundle E over a manifold M is a first order differential operator

$$\mathbb{A} : \Omega^*(M, E) \to \Omega^*(M, E),$$

such that

$$\mathbb{A}(\omega \otimes s) = d\omega \otimes s + (-1)^{deg\ \omega} \omega \otimes \mathbb{A}s. \qquad (\dagger)$$

Here $\Omega^*(M,E) = \Gamma(M, \bigwedge T^*M \otimes E)$ denotes the space of differential forms on M with values in E, ω is a form on M and s is a section of the bundle E. More abstractly, a superconnection can be interpreted as a d-derivation on the space of sections of a pull-back bundle on some supermanifold, denoted ΠTM. A supermanifold is like an ordinary manifold, except that it has more functions living on it, in particular "odd" functions which will anticommute, as opposed to the usual functions on a manifold, which commute. Quite often we will think of a supermanifold as just a $\mathbb{Z}/2$-graded algebra, representing the functions of the supermanifold. A typical example is the space of sections of an exterior algebra bundle (i.e. of the form $\bigwedge E^*$, for some vector bundle E) over an ordinary manifold. Then ΠTM is the supermanifold whose functions are the differential forms on the manifold M (i.e. $E = TM$ in the example above). If we agree that the functions on an "odd" vector space of dimension n form an exterior algebra on n generators, then we can call ΠTM the *odd tangent bundle* of M, i.e. the tangent bundle of M, with the fibers interpreted as "odd". On $\Omega^*(M)$, the space of functions on ΠTM, we have a standard (graded) derivation, namely the exterior derivative d. This can be interpreted as an *odd* vector field on ΠTM. Given a vector bundle E over M and a superconnection on E, we pull-back the bundle to ΠTM via a projection map $\pi : \Pi TM \to M$, and the superconnection is just a d-derivation in light of (†) above.

We will adopt here a different approach to superconnections, and view them as pairs consisting of a connection and an $\Omega^*(M)$-valued bundle endomorphism of E. This pair defines then a pair on the pull-back bundle via the map π consisting of the pull-back connection and an endomorphism of the pull-back bundle π^*E. Superpaths in M will then be lifted to superpaths in ΠTM and the parallel transport determined by the pull-back pair along the lifted superpaths will define the desired super-parallel transport of the superconnection. Before being able to do this, we need to understand what is the parallel transport associated to a connection on a super vector bundle over a supermanifold. The parallel transport is realized along superpaths, by solving some "half-order" differential equations, as opposed to the usual parallel transport along paths obtained by solving *first-order* differential equations. This half-integration process reduces by an algebraic manipulation to the usual integration process. We then add to the connection on a supervector bundle over a supermanifold a bundle endomorphism and obtain the version of parallel transport that is fit for understanding superconnections.

What is the motivation for such considerations? Well, the idea is that superconnections over a manifold M should provide interesting examples of supersymmetric 1-dimensional field theories over M. These examples are candidates for geometric cocycles in differential K-theory. Let's be more explicit. The main result of this paper is to produce out of a superconnection on a manifold M a notion of super-parallel transport based on M, which resembles closely the notion of a supersymmetric (susy, for short) 1-dim. field theory over M (see below for a rough definition of field theories). We think of superconnections as being in one-to-one correspondence with susy 1-dim field theories the same way as connections are in one-to-one correspondence to an appropriate notion of parallel transport (see [Poo81]). The transition is made via an integration process, and backwards, via a differentiation process. A superconnection should be just the infinitesimal version of a field theory, as a field theory should be the result of "integrating" a superconnection. Our aim is to define geometric cocycles for differential K-theory, so the hope is to have a sequence of equalities

$$\text{superconnections} \quad = \quad \text{susy 1-dim field theories} \quad = \quad \text{differential K-theory}.$$

If a cocycle in differential K-theory is indeed represented by a superconnection, we do have a map to K-theory, by forgetting the superconnection and remembering only the $\mathbb{Z}/2$-graded vector bundle, and a map to closed differential forms on M, by considering the Chern character form of the superconnection (see Section 2.5). Both the $\mathbb{Z}/2$-graded vector bundle and the Chern form will define the same class in the real cohomology of M. Superconnections therefore look like good candidates for differential K-theory.

From another point of view, the 1-dimensional case is a "study case" for the 2-dimensional field theories over M, and the hope is that these field theories would define geometric cocycles for elliptic cohomology, or rather for the "universal" elliptic cohomology, which is known as topological modular forms (tmf). (The name comes from the fact the cohomology theory tmf is constructed via an "inverse limit" over the category of elliptic curves, the same way as the *integral modular forms* are built. Moreover, there is an ring map from the coefficients of tmf to the ring of integral modular forms which is rationally an isomorphism.) Topological modular forms are well understood from a computational point of view (although a complete written account is not yet available in print- see for example M. Hopkins' ICM 2002 expository talk [Hop02], or the class notes of Ch. Rezk [Rez01]). A geometric understanding of TMF (the periodic version of tmf) is still missing although a lot of progress in this direction was made by Stolz-Teichner (see e.g.[ST04]). For alternate approaches to defining geometrically TMF see [HK04], [BDR04].

Elliptic cohomology arose in the mid '80s as a good place for invariants coming from field theories. The famous example is the Witten genus (see [Wit86]) for a string manifold, following the example of the elliptic genus introduced by Ochanine [Och87]. The Witten genus is nowadays refined to a map in homotopy theory $MU\langle 6\rangle \longrightarrow E$, for any elliptic spectrum E, called the *sigma orientation* map; see [AHS01], [AHS04]. At the same time, based on pioneering work of Atiyah [Ati88] and Segal [Seg88], field theories started being treated mathematically on their own right via an axiomatic approach which was meant to overcome the difficulties of path integration. In [Seg04], Segal suggests that conformal field theories (CFTs, for short) based on a manifold M might provide cocycles for the elliptic cohomology of M. The difficulty was that the field theories so defined couldn't constitute a cohomology theory. The missing ingredient was the Mayer-Vietoris property. Stolz and Teichner address this local problem and extend the definition of (2-dimensional) field theories one level down (i.e. to points). Another novelty in their approach is the introduction of supersymmetry. Their collaboration confirms the relationship between cohomology theories and field theories in the one dimensional case: the space of euclidean field theories (EFTs) is homotopy equivalent to $\Omega^\infty KO = \mathbb{Z} \times BO$, the 0^{th} space in the Ω-spectrum of KO-theory. (A similar statement holds in the complex case.) It is important that one considers here *supersymmetric* EFTs, otherwise the space of EFTs is contractible (see [ST04]). The next step would be to understand these field theories over parametrizing manifolds, and this is the direction for my thesis.

A d-dimensional field theory (see [Ati88], [Seg88]) gives a way to associate to a $(d-1)$-dimensional manifold a Hilbert space and to a bordism between such manifolds a trace class operator between the corresponding Hilbert spaces, in such a way that gluing of bordisms corresponds to composing operators. A field theory over a manifold M is as before except that the $(d-1)$-manifolds and the bordisms between them come equipped with maps to M. There are various flavors of field theories (topological, euclidean, conformal) according to what geometric structure one requires on bordisms (no structure, metric or conformal respectively). For a definition of supersymmetric euclidean field theories over a point see [ST04] (compare with [Mar05]). Field theories over a manifold can be thought of as describing some sort of parallel transport (along paths, modelling time, as is the case for EFTs, or conformal surfaces of arbitrary genus for CFTs- see e.g. [MP02] for a notion of parallel transport corresponding to gerbes) on the manifold, and the infinitesimal information is expected to be some kind of connection. If this data is enough to characterize field theories, the space of all such field theories will have a nice description. Equivalence classes of field theories are expected to give global information about the manifold, or, more optimistically, classes for (ordinary or differential) cohomology theories.

The paper is organized as follows. In Chapter 2 we review the basic theory of connections on principal bundles and vector bundles, and the relation between these. Parallel transport is introduced as a geometric interpretation of a connection. The exposition contains the topics that we want to generalize in the supercontext in the chapters to follow.

Chapter 3 is a brief introduction to supermanifolds and super Lie groups. Section 3.3 is dedicated to differential equations on supermanifolds. In particular, we show the existence and uniqueness of a flow of a vector field (even or odd) on a supermanifold.

In Chapter 4 we interpret geometrically connections on supervector bundles over supermanifolds as parallel transport along superpaths. This is realized by solving some half-order differential equations, as opposed to the usual parallel transport along paths, obtained by solving first-order differential equations. Another distinction from the usual parallel transport is that the transport is realized along *families* of superpaths. This idea of super parallel transport is a natural extension of the classical parallel transport as Section 4.3 will show. The usual properties of the parallel transport are present here as well. "Invariance under reparametrization" in this context means that the parallel transport is invariant under diffeomorphisms of superintervals that preserve the *conformal* structure (i.e. the distribution determined by the metric structure on superintervals, where metric structure means a nowhere vanishing odd vector field).

In the next chapter we deform the connection by a bundle endomorphism and see how the new parallel transport converges by an "inverse" adiabatic limit process to the parallel transport defined by the connection part only (in particular, the limiting parallel transport is invariant under reparametrization).

The last chapter contains our main result: any superconnection defines a super-parallel transport which in the inverse adiabatic limit is the super-parallel transport determined by the connection part of the superconnection. No two distinct super-connections give rise to the same super-parallel transport.

CHAPTER 2

THE STORY OF CONNECTIONS

In this section we will outline the theory of connections. The exposition is brief and for details the reader can consult the extensive literature on the subject. See for example [KN63] or Chapter 1 in [BGV92].

2.1 Connections on Principal Bundles

Let P be a principal G-bundle over a manifold M, with the group G acting from the right on the fibers. A *connection* on P is a choice of a horizontal invariant distribution $H \subset TP$ (i.e. for each $p \in P$ we have a splitting $T_pP = V_p \oplus H_p$ and, moreover, $R_{g*}H_p = H_{pg}$, for all $p \in P$, and all $g \in G$). Here V_p is the subspace of T_pP tangent to the fiber $P_{\pi(p)} = \pi^{-1}(\pi(p))$ and can be canonically identified with $\mathfrak{g}$, the Lie algebra of G, via

$$\mathfrak{g} \to V_p : X \mapsto \tilde{X} = \frac{d}{dt}\Big|_{t=0} p \cdot exp(tX).$$

Using this identification, we will not make much of a distinction between X and $\tilde{X}$. Note that, in particular, we obtain that, for each $p \in P$, $\pi_{*p} : H_p \to T_{\pi(p)}M$ is an isomorphism. Also, any vector field X on P can be written uniquely $X = vX + hX$, with vX (respectively hX) the vertical (respectively the horizontal) component of X.

An equivalent way to talk about a connection is as a *connection one-form*, that is, a form $\omega \in \Omega^1(P;\mathfrak{g})^G$ such that $i_{\tilde{X}}\omega = X$, for all $X \in \mathfrak{g}$. Explicitly, this means:

- $g^*\omega = Ad_{g^{-1}}\omega$, for all $g \in G$
- $\omega_p(\tilde{X}) = X$, for all $X \in \mathfrak{g}$.

To recover the old definition of a connection, we consider $H_p = \ker \omega_p$, for $p \in P$. An infinitesimal version of the first property above will give for any differential form $\alpha \in \Omega(P;\mathfrak{g})^G$ the relation

$$\mathfrak{L}_X\alpha + [X,\alpha] = 0, \text{ where } X \in \mathfrak{g}, \tag{2.1}$$

where $\mathfrak{L}_X$ is the *Lie derivative* in the direction of the vector field X.

Let (V,ρ) be a representation of G. A form $\alpha \in \Omega(P;V)$ is called *basic* if it is *horizontal*, i.e. $i_X\omega = 0$, for $X \in \mathfrak{g}$, and G-invariant, i.e. $g \cdot \alpha = \alpha$, for all $g \in G$. Here i_X is the *contraction* in the direction of X. The space of basic V-valued forms on P is denoted by $\Omega(P;V)_{bas}$.

Proposition 2.1. *There is an isomorphism*

$$\Omega(P;V)_{bas} \longrightarrow \Omega(M;P\times_\rho V)$$

$$\alpha \mapsto \alpha_M$$

with $\alpha_M(\pi_*X_1,\ldots,\pi_*X_q)(\pi(p)) = [p,\alpha(X_1,\ldots,X_q)(p)]$.

Proof. See [BGV92], Prop. 1.9. □

Proposition 2.2. *The space of connections on a principal G-bundle P over M is an affine space modelled on $\Omega^1(M;Ad(P))$, where $Ad(P) = P\times_{Ad}\mathfrak{g}$ is the adjoint bundle of P. Here Ad is the adjoint action of the Lie group G on its Lie algebra $\mathfrak{g}$.*

Proof. The difference of two connection 1-forms is a basic 1-form in $\Omega^1(P;V)_{bas}$. By the previous proposition, this space is identified with $\Omega^1(M;Ad(P))$. □

The *curvature* of a connection on P over M is the 2-form $\Omega \in \Omega^2(P; \mathfrak{g})$ defined by

$$\Omega(X, Y) = h[X, Y] - [hX, hY]$$

where X, Y are vector fields on P. Observe that Ω is a basic $\mathfrak{g}$-valued 2-form on P. It satisfies the Maurer-Cartan identity

$$\Omega = d\omega + [\omega, \omega]$$

where $[\omega, \omega](X, Y) = [\omega(X), \omega(Y)]$, with $[\ ,\ ]$ the Lie bracket of $\mathfrak{g}$. The curvature can be thought as the obstruction for the distribution H to be integrable.

2.2 Connections on Vector Bundles

Let E be a vector bundle on M. A *connection* on E (aka *covariant derivative*) is a differential operator

$$\nabla : \Gamma(M, E) \to \Gamma(M, T^*M \otimes E)$$

satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s.$$

In particular, for any vector field $X \in \mathcal{X}(\mathcal{M})$, we obtain an X-derivation

$$\nabla_X : \Gamma(M, E) \to \Gamma(M, E)$$

satisfying

$$\nabla_{fX} = f\nabla_X, \quad \nabla_{X+Y} = \nabla_X + \nabla_Y$$

$$[\nabla_X, f] = \text{multiplication by } X(f),$$

for $f \in \mathcal{C}^\infty(\mathcal{M}),\ \mathcal{X}, \mathcal{Y} \in \mathcal{X}(\mathcal{M})$.

For $v \in T_xM$ define a derivation $\nabla_v : \Gamma(M, E) \to E_x$ by

$$\nabla_v s := (\nabla_X s)(x), \text{ for some } X \in \mathcal{X}(M) \text{ with } X_x = v.$$

The definition is independent on the choice of X since the connection ∇ is $\mathcal{C}^\infty(M)$-*linear* in X.

The connection ∇ can be extended in a unique way to a map

$$\nabla : \Omega^*(M, E) \to \Omega^{*+1}(M, E)$$

by

$$\nabla(\alpha \wedge \theta) = d\alpha \wedge \theta + (-1)^{|\alpha|}\alpha \wedge \nabla\theta$$

for $\alpha \in \Omega(M),\ \theta \in \Omega(M, E)$. From the definition, if $\alpha \in \Omega^k(M, E)$ and $Y_0, Y_1, \dots, Y_k \in \mathcal{X}(\mathcal{M})$ then

$$(\nabla\alpha)(Y_0, \dots, Y_k) = \sum_{i=0}^{k}(-1)^i \nabla_{Y_i}(\alpha(Y_0, \dots, \hat{Y_i}, \dots, Y_k))$$
$$+ \sum_{i<j}(-1)^{i+j}\alpha([Y_i, Y_j], Y_0, \dots, \hat{Y_i}, \dots, \hat{Y_j}, \dots, Y_k).$$

The *curvature* R of a connection ∇ on E is the $End(E)$-valued 2-form:

$$R(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} \in \Gamma(M, End(E))$$

for $X, Y \in \mathcal{X}(\mathcal{M})$. R can be identified with $\nabla^2 = \nabla \circ \nabla : \Omega^*(M, E) \to \Omega^{*+2}(M, E)$.

2.3 Associated Vector Bundles and Associated Connections

Let (V, ρ) be a representation of G, i.e. $\rho : G \to Aut(V)$, and let $\rho_* = \rho_{*e} : \mathfrak{g} \to End(V)$ be the differential at the identity e of G. If $E = P \times_\rho V$ is the vector bundle associated to the representation (V, ρ) of G then a connection $\omega \in \Omega^1(P; \mathfrak{g})^G$ on P induces a connection ∇ on E as shown by the diagram

$$\begin{array}{ccc} \Gamma(M,E) & \xrightarrow{\nabla} & \Omega^1(M,E) \\ \cong\uparrow & & \cong\uparrow \\ \mathcal{C}^\infty(\mathcal{P};\mathcal{V})^{\mathcal{G}} & \xrightarrow{d+\rho(\omega)} & \Omega^1(P;V)_{bas} \end{array}$$

Here $\rho(\omega) = \rho_*(\omega) \in \Omega^1(P; End(V))$ and, multiplied by a function in $\mathcal{C}^\infty(P;V)$, produces an element in $\Omega^1(P;V)$. Another way to look at it is to say that for $s \in \Gamma(M,E) = \mathcal{C}^\infty(P;V)^G$, we let $\nabla s = ds \circ h$ viewed as an element of $\Omega^1(P,V)^G$, where h is the horizontal projection. Put differently, we have

$$\nabla_X s(x) = [p, d\tilde{s}(\tilde{X})] \tag{2.2}$$

where $\tilde{s}$ corresponds to s and $\tilde{X}$ is a (the) horizontal lift of $X \in \mathcal{X}(\mathcal{M})$, i.e. $\tilde{X} \in \Gamma(P,H)$ and $\pi_*\tilde{X} = X$.

Consider the map

$$\{\text{Connections on } P\} \xrightarrow{\Psi} \{\text{Connections on } E(P) = P \times_\rho V\}$$

which to a connection on P viewed for example as a horizontal distribution $H \subseteq TP$ associates the connection ∇ defined by (2.2). Then we have

Proposition 2.1. *(i) ρ_* injective $\Rightarrow \Psi$ injective.*
(ii) $G = Gl(n),\ V = \mathbf{R}^n \Rightarrow \Psi$ surjective.

Proof. (i) Consider a connection H on P and construct the associated connection ∇ by (2.2). We want to show that the horizontal distribution H can be recovered from ∇.

For that, let $\tilde{c} : \mathbf{R} \to P$ be a curve in P and let $c = \pi \circ \tilde{c}$ be its projection on M. For each $v \in V$ consider the curve $\tilde{c}_v$ in E, with

$$\tilde{c}_v(t) = [\tilde{c}(t), v] \in E_{c(t)}.$$

Claim:

$$\nabla_{\dot c(0)}\tilde c_v = [\tilde c(0), \rho_*(\nu\dot{\tilde c}(0))v] \in E_{c(0)},$$

where νw denotes the vertical component of a vector w.

Indeed,

$$\nabla_{\dot c(0)}[\tilde c(t), v] = [\tilde c(0), dv(h\dot{\tilde c}(0))] = [\tilde c(0), \rho_*(\nu\dot{\tilde c}(0))v].$$

The last equality goes as follows. Express $\tilde c_v(\cdot)$ as a "function" $\alpha : P \to V$ with $\alpha(\tilde c(t)g) = g^{-1}v$. We differentiate this function at $\tilde c(0)$ in the direction of $h\dot{\tilde c}(0)$. We first find a curve γ in P such that $\dot\gamma(0) = h\dot{\tilde c}(0)$. If $t \mapsto a_t$ generates $\nu\dot{\tilde c}(0)$ then $t \mapsto \tilde c(t)a_t^{-1}$ generates $h\dot{\tilde c}(0)$. Differentiating the identity $\alpha(\tilde c(t)a_t^{-1}) = a_t v$, we obtain the claim.

We can therefore say that $\dot{\tilde c}(0)$ is horizontal if

$$\nabla_{\dot c(0)}\tilde c_v = 0, \text{ for all } v \in V$$

Then

$$H_p = \{\dot{\tilde c}(0) \mid \dot{\tilde c}(0) \text{ horizontal}, \tilde c(0) = p\}.$$

(ii) Given a connection ∇ on E, we want to find a connection on P whose associated connection is ∇. Let us first note that the map

$$P \longrightarrow GL(E) = \{(x, f : \mathbf{R}^n \to E_x) | x \in M,\ f \text{ linear}\}$$

$$p \mapsto (x = \pi(p), f_p(e_i) = [p, e_i])$$

is a *canonical* isomorphism. Here $E_x = \{[p, v] |\ \pi(p) = x,\ v \in \mathbf{R}^n\}$ and $\{e_i\}$ is the standard basis of $\mathbf{R}^n$.

Let $\tilde c : \mathbf{R} \to P$ be a path in P, and c the corresponding path on M with $\dot c(0) \neq 0$. For each t we have $\tilde c(t) : \mathbf{R}^n \to E_{c(t)}$. For $v \in V = \mathbf{R}^n$, define $\tilde c_v(t) := \tilde c(t)v \in E_{c(t)}$.

We say that $\tilde{c}$ is horizontal (at $t = 0$) if

$$\nabla_{\dot{c}(0)} \tilde{c}_v = 0 \in E_{c(0)}, \text{all } v \in V.$$

Define

$$H_p = \{\dot{\tilde{c}}(0) \in T_p P \mid \tilde{c} \text{ horizontal (at } t = 0\text{)}\}.$$

Then $H \subset TP$ is a G-invariant distribution and $TP = VP \oplus H$, ie a connection on P. Indeed, let $\tilde{c}$ horizontal generating $\dot{\tilde{c}}(0)$. Then $\tilde{c}g$ generates $g_* \dot{\tilde{c}}(0)$. Moreover, $\tilde{c}g$ is horizontal at $t = 0$, since

$$\nabla_{\dot{c}(0)} (\tilde{c}g)_v = \nabla_{\dot{c}(0)} \tilde{c}(t) gv = 0, \text{ all } v \in V.$$

Also $\pi_* \dot{\tilde{c}}(0) = \dot{c}(0) \neq 0$. □

Corollary 2.2. *There is a 1-1 correspondence*

$$\{\textit{Connections on } P \cong GL(E)\} \xrightarrow{\Psi} \{\textit{Connections on } E\}.$$

We are now going to specify how to obtain new connections out of old ones. Let E be a vector bundle over M with connection ∇. We construct a connection on E^*, the dual bundle to E, as follows: for each X vector field on M and s^* section of E^*, define $\nabla_X s^*$ implicitly by the formula

$$(\nabla_X s^*, s) = X(s^*, s) - (s^*, \nabla_X s), \text{ for all } s \in \Gamma(M, E)$$

where $(\cdot, \cdot)$ denotes the natural pairing between E^* and E.

If E_1, E_2 are vector bundles over M with connections ∇_1, respectively ∇_2 then we can construct a connection ∇ on the tensor bundle $E_1 \otimes E_2$ by

$$\nabla = \nabla_1 \otimes 1 + 1 \otimes \nabla_2.$$

Let E be a vector bundle over M with connection ∇ and $\varphi : N \to M$ a smooth map. We can construct a connection $\varphi^*\nabla$ on the pull-back bundle φ^*E over N by

$$\varphi^*\nabla(\varphi^*s) := \varphi^*(\nabla s)$$

and extend it by the Leibniz rule. We call it the *pull-back* connection of ∇ via φ.

2.4 Connections and Parallel Transport

In this section, we will have another, more geometric, look at connections on bundles, as a way to identify the fibers of the bundle once we choose paths between their "base points". More precisely, we will describe the notion of parallel transport, and see how to recover the connection from the associated parallel transport.

Let us start with a vector bundle E over M with connection ∇, and consider a (piecewise) smooth path $c : \mathbf{R} \to M$ in M. (In the next chapters, when dealing with parallel transport along superpaths, we will always consider only smooth superpaths; see Section 4.5.) We obtain then a pull-back connection $c^*\nabla : \Gamma(\mathbf{R}, c^*E) \to \Gamma(\mathbf{R}, T^*\mathbf{R} \otimes c^*E)$. In particular, we obtain a derivation on the space of sections along the curve c, by considering $(c^*\nabla)_{\partial_t}$, denoted $\frac{\nabla}{dt}$. We say that a section $s \in \Gamma(\mathbf{R}, c^*E)$ is *parallel* if $\frac{\nabla}{dt}(s) = 0$. Let $[a, b] \subset \mathbf{R}$ and consider the curve c restricted to $[a, b]$, also denoted by c. We can define an isomorphism

$$\mathfrak{P}(c) : E_{c(a)} \to E_{c(b)} : v \mapsto s(b),$$

where s is the (unique) parallel section along c with $s(a) = v \in E_{c(a)}$. The existence and uniqueness of parallel sections follows from the theorem of existence and uniqueness of solutions of the system of first order differential equations of the type (2.3) below. The same theorem will also give that

$$\mathfrak{P}(c') \circ \mathfrak{P}(c) = \mathfrak{P}(c \cdot c') : E_{c(a)} \to E_{c'(d)}$$

where $c' : [b,d] \to M$, and $c \cdot c'$ is the juxtaposition of the paths c and c'.

Locally, in a trivializing neighborhood, the bundle $E \to M$ can be written $U \times V \to U$, with U open in M, and V a vector space of dimension the rank of E. Also, the connection is locally of the form $\nabla = d + \omega$, with d the standard derivation of functions with values in a vector space (in our case V) and $\omega \in \Omega^1(U, End E_{|U}) \cong \Omega^1(U) \otimes End(V)$. In this case, we have

$$(c^*\nabla)(c^*s) = c^*(\nabla s) = c^*(ds + \omega s) = d(s \circ c) + c^*\omega \cdot s \circ c$$

with s a local section of E. Therefore

$$\frac{\nabla}{dt}(s \circ c) = (c^*\nabla)_{\partial_t}(s \circ c) = \frac{d(s \circ c)}{dt} + \omega(\frac{dc}{dt})\ s \circ c. \tag{2.3}$$

This equation expresses the "parallel" condition in local coordinates.

Invariance under reparametrization. Let $\bar{c} = c \circ \varphi,\ t = \varphi(u)$, where $\varphi : \mathbf{R} \to \mathbf{R}$ is (piecewise) smooth. Then

$$(\bar{c}^*\nabla)_{\partial_u}(s \circ \bar{c}) = \frac{d(s \circ \bar{c})}{du} + \omega(\frac{d\bar{c}}{du})\ s \circ \bar{c} = \frac{d(s \circ c) \circ \varphi}{dt} \cdot \frac{d\varphi}{du} + \omega(\frac{dc}{dt} \circ \varphi \cdot \frac{d\varphi}{du}) s \circ c \circ \varphi.$$

In short,

$$\frac{\nabla}{du}(s \circ c \circ \varphi) = \frac{\nabla}{dt}(s \circ c) \circ \varphi \cdot \frac{d\varphi}{du}\ .$$

We also have that

$$\frac{\nabla}{du}(s \circ \varphi) = \frac{\nabla}{dt}(s) \circ \varphi \cdot \frac{d\varphi}{du} \tag{2.4}$$

for s an *arbitrary* section of the pull-back bundle c^*E. Thus a section s is parallel along c if and only if $s \circ \varphi$ is parallel along $\bar{c} = c \circ \varphi$. Therefore $\mathfrak{P}$ is invariant under reparametrization:

$$\mathfrak{P}(c) = \mathfrak{P}(c \circ \varphi), \qquad \text{with } \varphi : [a', b'] \to [a, b],\ \varphi(a') = a,\ \varphi(b') = b.$$

To summarize, a connection ∇ on the bundle E over M gives rise to a map $\mathfrak{P}$ defined on the space of paths in M,

$$\mathcal{P}(M) := \{ \text{ paths } c : [a, b] \to M\}$$

satisfying

(i) $\mathfrak{P}(c_x) = 1_{E_x}$, where c_x is a constant path at x.

(ii) $\mathfrak{P}(c') \circ \mathfrak{P}(c) = \mathfrak{P}(c \cdot c')$

(iii) $\mathfrak{P}(c \circ \varphi) = \mathfrak{P}(c)$.

(iv) $\mathfrak{P}(c)$ is invertible, with inverse given by $\mathfrak{P}(c)^{-1} = \mathfrak{P}(c^{-1})$.

We call a map $\mathfrak{P}$ as above *parallel transport* on E.

The correspondence

$$c \longmapsto \mathfrak{P}(c)$$

determined by a connection ∇ on E is *smooth* in the following sense:

Let $I \to S$ be a family of intervals parametrized by the manifold S. Such an I is given by the triplet

$$S \xhookrightarrow[1\times a]{} S \times \mathbf{R} \xhookleftarrow[1\times b]{} S\ ,$$

with $a, b : S \to \mathbf{R}$ smooth maps such that $a < b$, by "deleting" the part of $S \times \mathbf{R}$ to the left of a and to the right of b. Such a family is denoted by $I^{a,b}$. Given a smooth family of curves in M, $c : I^{a,b} \to M$ parametrized by S, the parallel transport gives rise to a *smooth* bundle map

$$\begin{array}{ccc} c(a)^*E & \xrightarrow{\mathfrak{P}(c)} & c(b)^*E \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

Instead of working with vector bundles we could as well work with principal bundles. Let, therefore, P be a G-bundle over M with connection $H \subset TP$, and let

$c : [a, b] \to M$ a curve in M. Fix $p \in P_{c(a)}$. Then (see [KN63], p. 69) there is exactly one horizontal curve $\tilde{c} : [a, b] \to P$ which is a lift of c and such that $\tilde{c}(a) = p$. Define then an isomorphism, which is G-invariant,

$$\mathfrak{P}(c) : P_{c(a)} \to P_{c(b)} : p \mapsto \tilde{c}(b)$$

where $\tilde{c}$ is as above. The map $\mathfrak{P}$ will satisfy the properties of parallel transport, as before.

Given a connection ∇ on E and the associate parallel transport $\mathfrak{P}$, we can use it to *recover* the connection. More precisely, the following holds.

Proposition 2.1. *Let E and ∇ be as before, with associated $\mathfrak{P}$. Then,*

$$\nabla_v s = \lim_{h \to 0} \frac{\mathfrak{P}(c_h^{-1}) s(c(h)) - s(x)}{h}$$

with $v \in T_x M$, s a section in a neighborhood of $x \in M$, c a curve in M with $\dot{c}(0) = v$, and c_h its restriction to $[0, h]$.

Proof. (see [Hel01]) Let U be a trivializing neighborhood of x. Then $E_{|U} \cong U \times \mathbf{R}^n$, with n the rank of E. Let $\{\frac{\partial}{\partial x_i}\}$ be a basis of vector fields in U and $\{s_j\}$, $j = 1, \ldots, n$ linearly independent sections of E over U. Denote $\omega(\frac{\partial}{\partial x_i}) s_j = \Gamma_{ij}^k s_k$, with Γ_{ij}^k the usual Christoffel symbols. Here we use the usual Einstein summation convention, that repeated indices are to be summed. Let $y_h(t)$ be the section parallel along $c(t)$ with $y_h(h) = s(c(h))$. Then, by (2.3), we obtain the system

$$\begin{cases} \frac{dy_h^k}{dt} + \Gamma_{ij}^k \frac{dc^i}{dt} y_h^j = 0 \\ \\ y_h^k(h) = s^k(c(h)), \qquad k = 1, \ldots, n \end{cases}$$

By Taylor's theorem, $y_h^k(h) = y_h^k(0) + h\frac{dy_h^k}{dt}(t^*)$, for some $t^* \in (0, h)$. Then

$$\begin{aligned} RHS &= \lim_{h\to 0} \frac{y_h^k(0) - s^k(0)}{h} \\ &= \lim_{h\to 0} \Big[\frac{y_h^k(h) - s^k(0)}{h} - \frac{dy_h^k}{dt}(t^*)\Big] \\ &= \lim_{h\to 0} \Big[\frac{s^k(c(h)) - s^k(0)}{h} + \Gamma_{ij}^k(t^*)\frac{dc^i}{dt}(t^*)y_h^j(t^*)\Big] \\ &= \nabla_v s. \end{aligned}$$

□

Remark 2.2. *With an appropriate notion of parallel transport on a vector bundle E over a manifold, one can build a connection on E whose parallel transport is the one given. For details, the reader can consult [Poo81].*

2.5 Quillen Connections

Let E be a $\mathbb{Z}/2$-graded vector bundle over a smooth manifold M. Consider the space of E-valued differential forms on M,

$$\Omega^*(M, E) = \Omega^*(M) \otimes_{C^\infty(M)} \Gamma(M, E).$$

This is a $\mathbb{Z}/2$-graded vector space, with the grading induced from the grading on forms on M by even and odd forms, and the grading on E. A *(Quillen) connection*, or a *superconnection* is an odd derivation

$$\mathbb{A} : \Omega^*(M, E) \to \Omega^*(M, E)$$

such that

$$\mathbb{A}(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^{|\omega|}\omega \wedge \mathbb{A}\alpha,$$

for $\omega \in \Omega^*(M)$ and $\alpha \in \Omega^*(M, E)$. A Quillen connection $\mathbb{A}$ can be written:

$$\mathbb{A} = \mathbb{A}_0 + \mathbb{A}_1 + \mathbb{A}_2 + \dots$$

where the i^{th} piece $\mathbb{A}_i$ maps $\Omega^k(M, E)$ into $\Omega^{k+i}(M, E)$.

Remarks.

1. $\mathbb{A}$ is determined by restricting it to $\Gamma(M, E) = \Omega^0(M, E)$ and using the derivation property.

2. $\mathbb{A}_1$ is a covariant derivative preserving the grading.

3. $\mathbb{A}_i$, for $i \neq 1$, acts by multiplication by a form, denoted also $\mathbb{A}_i$, in $\Omega^i(M, EndE)$

4. The space of Quillen connections on E is an affine space modelled on $\Omega^*(M, EndE)^{odd}$.

5. If $\mathbb{A}$ and $\mathbb{B}$ are Quillen connections on the vector bundle E, respectively F then $\mathbb{A} \otimes 1 + 1 \otimes \mathbb{B}$, defined by

$$(\mathbb{A} \otimes 1 + 1 \otimes \mathbb{B})(\alpha \otimes \beta) = \mathbb{A}\alpha \otimes \beta + (-1)^{|\alpha|}\alpha \otimes \mathbb{B}\beta$$

   is a Quillen connection on $E \otimes F$.

6. Let $\varphi : N \to M$ be a smooth map and E a $\mathbb{Z}/2$-graded vector bundle on M with Quillen connection $\mathbb{A}$. Then $\varphi^*\mathbb{A}$ is a Quillen connection on φ^*E over N defined by requiring

$$\varphi^*\mathbb{A}(\varphi^* s) = \varphi^*(\mathbb{A}s), \qquad s \in \Omega(M, E).$$

   This is called the *pullback* of $\mathbb{A}$ by φ.

To a bundle over a base manifold M, one can associate certain *characteristic classes* in the cohomology of the base. Moreover, a choice of a (Quillen) connection, allows one to represent these classes by differential forms, via deRham cohomology.

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}/2$-graded vector space and let $A \in End(V)$ be an endomorphism of V. Define the *supertrace* of $A = (a_{ij})_{0 \leq i,j \leq 1}$ by $str(A) = tr\ a_{00} - tr\ a_{11}$, where $a_{ij} \in Hom(V_i, V_j)$. For a $\mathbb{Z}/2$-graded vector bundle E over M define the supertrace map

$$str : \Gamma(M, End(E)) \to C^\infty(M) : (a_{ij})_{0 \leq i,j \leq 1} \mapsto tr\ a_{00} - tr\ a_{11},$$

with $a_{ij} \in \Gamma(M, Hom(E_i, E_j))$. The supertrace str can be extended uniquely to a map

$$str : \Omega(M, End(E)) \to \Omega(M) : \alpha \otimes A \mapsto \alpha\ str(A).$$

Let $f \in \mathbf{C}[[z]]$ be a power series in z. Let $\mathbb{A}^2 \in \Omega^2(M, End\ E)$ be the curvature of a Quillen connection $\mathbb{A}$ on E. Consider

$$f(\mathbb{A}^2) = \sum \frac{f^{(n)}(0)}{n!}(\mathbb{A}^2)^n$$

as an element of $\Omega^{ev}(M, End\ E)$. Then (see [BGV92], Prop. 1.41), we have

Proposition 2.1. *(i) $str\ f(\mathbb{A}^2)$ is a closed differential form.*

(ii) Let $\mathbb{A}_t$ be a 1-parameter family of Quillen connections. Then

$$\frac{d}{dt} str\ f(\mathbb{A}_t^2) = d\ str\left(\frac{d\mathbb{A}_t}{dt} f'(\mathbb{A}_t^2)\right).$$

Remark 2.2. *As a consequence of (ii), we have that $str f(\mathbb{A}^2)$ defines a cohomology class in $\Omega(M)$, independently of the choice of connection $\mathbb{A}$.*

Examples.

1. Consider $f(z) = e^{-z}$. Define then

$$ch(\mathbb{A}) = str(e^{-\mathbb{A}^2}),$$

called the *Chern character* form of $\mathbb{A}$. It is additive w.r.t. direct sums of connections and multiplicative w.r.t. tensor products. Its cohomology class is denoted $ch(E)$.

2. To a real vector bundle with connection ∇ we associate the form

$$\hat{A}(\nabla) = det^{1/2}\Big(\frac{\nabla^2/2}{\sinh \nabla^2/2}\Big) \in \Omega^{4*}(M, \mathbf{R})$$

called the $\hat{A}$-genus form of ∇. It is multiplicative in the sense that

$$\hat{A}(\nabla_1 \oplus \nabla_2) = \hat{A}(\nabla_1) \wedge \hat{A}(\nabla_2).$$

Via the formula $det\ A = \exp(tr\ \log A)$, it can be thought of as the exponential of the characteristic form defined by the power series $f(z) = \frac{1}{2} \log \frac{z/2}{\sinh z/2}$. Its cohomology class is denoted by $\hat{A}(E)$.

3. Similarly, we can define

$$Td(\nabla) = det\Big(\frac{\nabla^2}{e^{\nabla^2} - 1}\Big) \in \Omega^{2*}(M, \mathbf{C}),$$

called the Todd genus form of ∇. The Todd genus is also multiplicative, and can be viewed as the exponential characteristic class associated to the series $f(z) = \log \frac{z}{e^z - 1}$. Its cohomology class is denoted by $Td(E)$.

CHAPTER 3

SUPERMANIFOLDS

This chapter gives a brief introduction to the theory of supermanifolds. The subject was introduced and developed by Leites [Lei80] , Bernstein, Manin [Man88]. The standard reference is Deligne and Morgan [DM99], along with Varadarajan [Var04]. The section on super Lie groups and group actions follows closely the exposition in [BBHR91].

3.1 Generalities on Supermanifolds

A *supermanifold* M of dimension $p|q$ is a pair $(|M|, \mathcal{O}_M)$ with $|M|$ a smooth manifold of dimension p and $\mathcal{O}_M$ is a sheaf of $\mathbf{Z}/2$-graded algebras that looks, locally on $|U| \subseteq |M|$, like $\mathcal{C}^\infty(|U|) \otimes \Lambda[\theta_1, \ldots, \theta_q]$. Roughly speaking, a supermanifold is an ordinary manifold with the space of functions enriched to contain "odd" functions as well. $|M|$ is the *underlying space* of M and $\mathcal{O}_M$ is the *structure sheaf* of M. The odd functions generate a nilpotent ideal $\mathcal{J}$ of $\mathcal{O}_M$ and $(|M|, \mathcal{O}_M/\mathcal{J})$ is a smooth manifold of dimension p, called the *reduced* manifold M_{red} of M. *Open submanifolds* U of M are open subspaces $|U|$ of $|M|$ with structure sheaf the restriction of $\mathcal{O}_M$ to $|U|$.

E.g. $\mathbf{R}^{p|q} = (\mathbf{R}^p, \mathcal{O}_{\mathbf{R}^{p|q}})$ with $\mathcal{O}_{\mathbf{R}^{p|q}} = \mathcal{C}^\infty_{\mathbf{R}^p} \otimes \Lambda[\theta_1, \ldots, \theta_q]$; open submanifolds in $\mathbf{R}^{p|q}$. These are the local models for the supermanifolds therefore we can think of them as *affine* supermanifolds.

A *morphism* $f : M \to N$ is a pair $f = (|f|, f^\sharp)$ consisting of a continuous map $|f| : |M| \to |N|$ and a map $f^\sharp : \mathcal{O}_N \to |f|_* \mathcal{O}_M$ of sheaves of $\mathbb{Z}/2$-graded algebras. For example, there is a canonical morphism

$$i : M_{red} \hookrightarrow M,$$

which on the underlying spaces is the identity and the map on sheaves is the projection $i^\sharp : \mathcal{O}_M \to \mathcal{O}_M/\mathcal{J}$. A morphism $f : M \to N$ induces a morphism between the corresponding reduced manifolds since it preserves the nilpotent ideal sheaves.

Supermanifolds can be thought as particular examples of ringed spaces. The category **SM** of supermanifolds embeds fully faithfully in the category of ringed spaces.

An important source of examples of supermanifolds comes from vector bundles. To any vector bundle E over a manifold M_0 one can associate a supermanifold $\Pi E = (M_0, \mathcal{O}_{\Pi E})$ where $\mathcal{O}_{\Pi E}$ is the sheaf of sections of ΛE^*. This defines a functor:

$$S : \mathbf{VB} \to \mathbf{SM} : E \mapsto \Pi E.$$

There is also a functor V going the other direction. Namely, let $M = (M_0, \mathcal{O}_M)$ be a supermanifold. Then $(\mathcal{J}/\mathcal{J}^2)^*$ is a locally free sheaf on M_0, where $\mathcal{J}$ is the nilpotent ideal of $\mathcal{O}_M$, so it determines a vector bundle on M_0. We have that $V \circ S = id$ and $S \circ V = id$ on isomorphism classes of objects. This doesn't assure an equivalence of categories though, since SV fails to be the identity on morphisms (e.g. it maps the automorphism $(x, \theta_1, \theta_2) \mapsto (x + \theta_1\theta_2, \theta_1, \theta_2)$ of $\mathbf{R}^{1|2}$ to id). The category of supermanifolds is richer in morphisms. This relation between the categories is analogous to the one between graded rings (vector bundles) and filtered rings (supermanifolds).

There is another useful way to describe morphisms in **SM** between supermanifolds and affine supermanifolds (open in $\mathbf{R}^{p|q}$). Let $f : M \to U$ with U affine. Then f gives a map of sheaves $f^\sharp : \mathcal{O}_U \to |f|_* \mathcal{O}_M$. Taking global sections, we obtain a natural map:

$$\mathbf{SM}(M, U) \to \mathbf{Alg}(\mathcal{O}_U(|U|), \mathcal{O}_M(|M|)).$$

This map is bijective. Thus, to give a map between a supermanifold and an affine supermanifold is enough to give an algebra map of their global sections. It also says that to give a map from M to $U \subseteq \mathbf{R}^{p|q}$ is to give p even functions $f_1, \ldots, f_p$ and q odd functions $g_1, \ldots, g_q$ on $|M|$ such that $(f_1, \ldots, f_p)(|M|) \subseteq |U|$. They are obtained by looking at the standard coordinate functions on $U \subseteq \mathbf{R}^{p|q}$. In this manner, we can make use of coordinates on supermanifolds.

Remark 3.1. *In general, maps between two supermanifolds are uniquely determined by the map induced on global sections (see [Kos77], p. 208). That's why we quite often write a map between sheaves as just the map induced on their global sections.*

The tangent sheaf and tangent vectors. The analogue of the tangent bundle in classical differential geometry is the *tangent sheaf* $\mathcal{T}M$ defined as the sheaf of graded derivations of $\mathcal{O}_M$, i.e. for $U \subseteq |M|$

$$\mathcal{T}M(U) = \{X : \mathcal{O}(U) \to \mathcal{O}(U) \text{ linear } | X(fg) = X(f)g + (-1)^{p(X)p(f)} f X(g)\}.$$

Here $p(X) = 0$ or 1 according to whether X is even, respectively odd vector field on U, and similarly $p(f) = 0$ or 1, for f even, respectively odd, function on M. $\mathcal{T}M$ is then a locally free $\mathcal{O}_M$-module of rank (p, q) the dimension of the supermanifold M. Sections of $\mathcal{T}M$ are the *vector fields* on M. For X and Y vector fields on M, define as usual their *Lie bracket* $[X, Y]$ by

$$[X, Y](f) = X(Y(f)) - (-1)^{p(X)p(Y)} Y(X(f)), \text{ for } \quad f \in \mathcal{C}^\infty(M) = \mathcal{O}_M(|M|).$$

For example, consider on $\mathbf{R}^{1|1}$ the vector field $D = \partial_\theta + \theta\partial_t$. Then, one can check that

$$D^2 = \frac{1}{2}[D, D] = \partial_t.$$

Similarly, if $Q = \partial_\theta - \theta\partial_t$, then

$$Q^2 = \frac{1}{2}[Q, Q] = -\partial_t.$$

Let (x^i, θ^j) be local coordinates in a local neighborhood $U \subseteq M$ of the point $m \in M$ (i.e. x^i, θ^j are the the pull-back of the canonical coordinates on $V \subseteq \mathbf{R}^{p|q}$, via an isomorphism $\psi : U \to V$), then

$$\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \theta^j}\right\}$$

forms a basis of $\mathcal{T}M(U)$ over $\mathcal{O}(U)$. Here:

$$\frac{\partial}{\partial x^i}(\sum_I f_I\theta^I) = \sum_I \frac{\partial f_I}{\partial x^i}\theta^I, \qquad \frac{\partial}{\partial \theta^j}(\sum_{j\notin I} f_I\theta^I + \sum_{j\notin I} f_{j,I}\theta^j\theta^I) = \sum_{j\notin I} f_{j,I}\theta^I.$$

At a point $m \in M$ (i.e. $m \in |M|$), a (graded) *tangent vector* is a derivation $\mathcal{O}_{M,m} \xrightarrow{v} \mathbf{R}$, i.e. a linear map v such that:

$$v(fg) = v(f)g(m) + (-1)^{p(v)p(f)}f(m)v(g).$$

Here $\mathcal{O}_{M,m}$ is the stalk at m. In local coordinates (x^i, θ^j), the tangent vector v can be written as a linear combination of $(\partial/\partial x^i)_m$ and $(\partial/\partial\theta^j)_m$, with $i = 1, \ldots, p,\ j = 1, \ldots, q$. The tangent space at $m \in M$, denoted TM_m, is thus a super vector space of dimension $p|q$. If $f : M \to N$ is a map, and $m \in M$, then the *differential* of f at m is the morphism $df_m : TM_m \to TN_{f(m)} : v \mapsto v \circ f^\sharp$.

More generally, for $m \in M(S)$ an S-point of M (see below), define the tangent space at m to M by

$$TM_m = \{v : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(S) |\ v(fg) = v(f)m^\sharp(g) + (-1)^{p(v)p(f)} m^\sharp(f)v(g)\}.$$

The cotangent sheaf and exterior cotangent sheaf. Define the *cotangent* sheaf Ω^1_M to be the dual of the tangent sheaf $\mathcal{T}M$. Sections of Ω^1_M are called differential 1-forms. Let $<\ ,\ >: \mathcal{T}M \times \Omega^1_M \to \mathcal{O}_M$ denote the duality pairing between vector fields and 1-forms. Define the *exterior derivative* $d : \mathcal{O}_M \to \Omega^1_M$ by

$$< X, df >= X(f), \quad \text{for } X \in \mathcal{T}M, \ \ f \in \mathcal{O}_M.$$

Let $\Omega^*_M = \Lambda^* \Omega^1 M$ be the *exterior cotangent* sheaf on M, whose sections are called exterior differential forms on M. Then d extends uniquely to a grading preserving derivation $d : \Omega^*_M \to \Omega^*_M$ by requiring that

$$d^2 = 0,$$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta, \text{ for } \alpha \in \Omega^p_M.$$

For more details, the reader can consult [DM99], p. 73.

Geometric structures on $(1,1)$-supermanifolds. Let Y be a $(1,1)$-supermanifold. Then, the tangent sheaf $\mathcal{T}Y$ is a locally free $\mathcal{O}_Y$-module of rank (1,1): if (t, θ) are local coordinates on Y, then $\{\partial_t,\ \partial_\theta\}$ form a local basis for $\mathcal{T}Y$. A *conformal structure* on Y is a rank (0,1)-distribution $\mathcal{D}$, i.e. a rank (0,1) subsheaf of the tangent sheaf $\mathcal{T}Y$, that fits into the following short exact sequence of sheaves

$$0 \to \mathcal{D} \longrightarrow \mathcal{T}Y \longrightarrow \mathcal{D}^{\otimes 2} \to 0.$$

A *euclidean (metric) structure* on Y is given by the choice of an odd vector field D generating an odd distribution $\mathcal{D}$ as above. For example, on $\mathbf{R}^{1|1}$ consider the vector field $D = \partial_\theta + \theta\partial_t$. Then D defines a metric structure on $\mathbf{R}^{1|1}$, called the *standard* metric structure on $\mathbf{R}^{1|1}$. Also $\mathcal{D} =< D >$, the distribution generated by D, defines a conformal structure on $\mathbf{R}^{1|1}$: indeed, the square of D is $D^2 = \partial_t$, and the pair $\{D,\ D^2\}$ generates $\mathcal{T}\mathbf{R}^{1|1}$ as an $\mathcal{O}_{\mathbf{R}^{1|1}}$-module. For an alternative definition of metric structures, see [ST04], section 3.2.

The "functor of points" viewpoint. In the superworld one cannot talk properly about points on a supermanifold unless one refers to points on the reduced manifold. A more suitable approach is the lingo of *S-points*. Consider M a supermanifold. An S-point of M for an arbitrary supermanifold S is a map $S \to M$ and the S-points of M is the *set* $M(S) = \mathbf{SM}(S, M)$. This is the approach physicists adopt in computations, which also resonates with our geometric intuition. One can think of an S-point as a family of points of M parametrised by S. For example, as sets:

$$M(\mathbf{R}^{0|0}) = \mathbf{SM}(\mathbf{R}^{0|0}, M) = M_{red}.$$

If $T \stackrel{\alpha}{\longrightarrow} S$ is a map in $\mathbf{SM}$, there is a natural map $M(S) \to M(T) : m \mapsto m \circ \alpha$. So M determines a contravariant functor:

$$\mathbf{SM}^{op} \to \mathbf{Sets} : S \mapsto M(S)$$

called the *functor of points* of M. A map $f : M \to N$ of supermanifolds determines a natural transformation $\mathbf{SM}(\ \cdot\ , M) \to \mathbf{SM}(\ \cdot\ , N)$. The converse of this is also true, and forms the content of Yoneda's lemma. This means that to give a map $M \to N$ amounts to giving maps of sets $M(S) \to N(S)$, natural in S. This point of view will be especially useful in the next section.

One can therefore think of a supermanifold M as a representable functor $\mathbf{SM}^{op} \to \mathbf{Sets}$, such a functor determining M uniquely up to isomorphism. For example, if M, N are two supermanifolds, their *product* $M \times N$ can be interpreted as the supermanifold representing the functor

$$S \mapsto \mathbf{SM}(S, M) \times \mathbf{SM}(S, N).$$

An arbitrary contravariant functor

$$\mathbf{SM} \to \mathbf{Sets}$$

will be called a *generalized supermanifold*. The category **SM** of supermanifolds embeds fully faithful into the category **GSM** of generalized supermanifolds. Consider, for example, two supermanifolds M, N and define the generalized supermanifold

$$\underline{\mathbf{SM}}(M, N): \ \mathbf{SM} \to \mathbf{Sets} : S \longmapsto \mathbf{SM}(S \times M, N).$$

If $\underline{\mathbf{SM}}(M, N)$ is an ordinary supermanifold, then we have the following adjunction formula

$$\mathbf{SM}(S, \ \underline{\mathbf{SM}}(M, N)) \cong \mathbf{SM}(S \times M, N).$$

Lemma 3.2. *Let M be an ordinary manifold. Then, we can identify*

$$\underline{\mathbf{SM}}(\mathbf{R}^{0|1}, M) \cong \Pi TM,$$

where ΠTM is the odd tangent bundle of M.

Proof. We want to show that we have isomorphisms

$$\Psi_S : \mathbf{SM}(S \times \mathbf{R}^{0|1}, M) \to \mathbf{SM}(S, \Pi TM),$$

natural in S, where S is an arbitrary supermanifold. The left hand side is, via Remark 3.1, the set of grading preserving maps of $\mathbb{Z}/2$-algebras

$$\varphi : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(S \times \mathbf{R}^{0|1}) = \mathcal{C}^\infty(S)[\theta].$$

If we write $\varphi(f) = \varphi_1(f) + \theta\varphi_1(f)$, for $f \in \mathcal{C}^\infty(M)$, then the fact that $\varphi(fg) = \varphi(f)\varphi(g)$ is equivalent to the following two conditions

$$\begin{cases} \varphi_1(fg) = \varphi_1(f)\varphi_1(g) \\ \varphi_2(fg) = \varphi_2(f)\varphi_1(g) + (-1)^{p(f)}\varphi_1(f)\varphi_2(g) \end{cases}$$

The first condition is equivalent to $\varphi_1 = a^\sharp$, for some $a : S \to M$. The second condition tells us that φ_2 is an *odd* tangent vector at $a \in M(S)$, i.e. $\varphi_2 = X_a \in TM_a$. (For the definition of a tangent vector at an S-point, see the paragraph on tangent vectors above.) Therefore the left hand side is

$$\mathbf{SM}(S \times \mathbf{R}^{0|1}, M) = \ \{\text{pairs } (a, X_a) \mid a \in M(S),\ X_a \in TM_a,\ X_a \text{ odd}\}.$$

The right hand side $\mathbf{SM}(S, \Pi TM)$ is the set of $\mathbb{Z}/2$-graded algebra maps $\Omega^*(M) \to \mathcal{C}^\infty(S)$. Such maps are determined by their restriction to 0-forms (functions) and 1-forms (more specifically, 1-forms of the type df, for $f \in \mathcal{C}^\infty(M)$). Define then $\Psi_S(a, X_a)$ to be the map $S \to \Pi TM$ determined by defining it on functions $f \in \mathcal{C}^\infty(M)$ by $a^\sharp(f) \in \mathcal{C}^\infty(S)$, and on forms df by $X_a(f)$. One can easily check that Ψ_S is well-defined, bijective, and natural in S.

□

Let $T : \Pi TM \times \mathbf{R}^{0|1} \to \Pi TM$ be the map which on functions is given by $\Omega^*(M) \ni \omega \mapsto \omega + \theta d\omega \in \Omega^*(M)[\theta]$. Consider also the map $\mu : \underline{\mathbf{SM}}(\mathbf{R}^{0|1}, M) \times \mathbf{R}^{0|1} \to \underline{\mathbf{SM}}(\mathbf{R}^{0|1}, M)$, defined on S-points

$$\mathbf{SM}(S \times \mathbf{R}^{0|1}, M) \times \mathbf{SM}(S, \mathbf{R}^{0|1}) \to \mathbf{SM}(S \times \mathbf{R}^{0|1}, M)$$

by $(\varphi, \eta) \mapsto \varphi \circ (1 \times m) \circ (1 \times \eta \times 1) \circ (\Delta \times 1)$, where m is the composition map on $\mathbf{R}^{0|1}$. The maps T and μ define an action of $\mathbf{R}^{0|1}$ on the corresponding spaces.

Lemma 3.3. *The map defined in the previous lemma*

$$\Psi : \underline{\mathbf{SM}}(\mathbf{R}^{0|1}, M) \to \Pi TM$$

is $\mathbf{R}^{0|1}$-equivariant.

Proof. We want to show that the following diagram is commutative

$$\begin{array}{ccc} \underline{\mathbf{SM}}(\mathbf{R}^{0|1}, M) \times \mathbf{R}^{0|1} & \xrightarrow[\Psi \times 1]{} & \Pi TM \times \mathbf{R}^{0|1} \\ \mu \downarrow & & \downarrow T \\ \underline{\mathbf{SM}}(\mathbf{R}^{0|1}, M) & \xrightarrow[\Psi]{} & \Pi TM. \end{array}$$

We need that, for each supermanifold S, natural in S, the following diagram commutes

$$\begin{array}{ccc} \mathbf{SM}(S \times \mathbf{R}^{0|1}, M) \times \mathbf{SM}(S, \mathbf{R}^{0|1}) & \xrightarrow[\Psi_S \times 1]{} & \mathbf{SM}(S, \Pi TM) \times \mathbf{SM}(S, \mathbf{R}^{0|1}) \\ \mu_S \downarrow & & \downarrow T_S \\ \mathbf{SM}(S \times \mathbf{R}^{0|1}, M) & \xrightarrow[\Psi_S]{} & \mathbf{SM}(S, \Pi TM), \end{array}$$

or, in terms of functions we need to have

$$\begin{array}{ccc} \mathbf{Alg}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(S)[\theta]) \times \mathcal{C}^\infty(S)^{odd} & \xrightarrow[\Psi_S \times 1]{} & \mathbf{Alg}(\Omega^*(M), \mathcal{C}^\infty(S)) \times \mathcal{C}^\infty(S)^{odd} \\ \mu_S \downarrow & & \downarrow T_S \\ \mathbf{Alg}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(S)[\theta]) & \xrightarrow[\Psi_S]{} & \mathbf{Alg}(\Omega^*(M), \mathcal{C}^\infty(S)). \end{array}$$

For $a \in M(S)$ and $X_a \in TM_a$ denote by $(a, X_a) \in \mathbf{Alg}(\Omega^*(M), \mathcal{C}^\infty(S))$ the map determined by $f \mapsto a^\sharp(f)$ and $df \mapsto X_a(f)$. (Compare the proof of the previous lemma.) Via the identification

$$\mathbf{Alg}(\Omega^*(M), \mathcal{C}^\infty(S)) \times \mathcal{C}^\infty(S)^{odd} = \mathbf{Alg}(\Omega^*(M)[\theta], \mathcal{C}^\infty(S)),$$

the map

$$T_S : \mathbf{Alg}(\Omega^*(M), \mathcal{C}^\infty(S)) \times \mathcal{C}^\infty(S)^{odd} \to \mathbf{Alg}(\Omega^*(M), \mathcal{C}^\infty(S))$$

evaluated at $\varphi = ((a, X_a), \tilde{\theta}) \in \mathbf{Alg}(\Omega^*(M), \mathcal{C}^\infty(S)) \times \mathcal{C}^\infty(S)^{odd}$ is determined by saying that

$f \xmapsto{T^\sharp} f + \theta dt \xmapsto{\varphi^\sharp} a^\sharp(f) + \tilde{\theta} X_a(f) =: b^\sharp(f) \quad$ and

$df \xmapsto{T^\sharp} df \xmapsto{\varphi^\sharp} X_a(f) =: X_b(f),$

where $b \in M(S)$ is defined by $b^\sharp(f) = a^\sharp(f) + \tilde{\theta} X_a(f)$, for $f \in \mathcal{C}^\infty(M)$.

On the other hand,

$$\mu_S : \mathbf{SM}(S \times \mathbf{R}^{0|1}, M) \times \mathbf{SM}(S, \mathbf{R}^{0|1}) \to \mathbf{SM}(S \times \mathbf{R}^{0|1}, M)$$

is defined by

$$(\alpha = (a, X_a), \eta) \longmapsto \alpha \circ (1 \times m) \circ (1 \times \eta \times 1) \circ (\Delta \times 1),$$

or, on functions, $\mu_S((a, X_a), \tilde{\theta})$ is given by

$$\begin{aligned} f \quad & \xmapsto{\alpha^\sharp} \quad a^\sharp(f) + \theta X_a(f) \\ & \xmapsto{1 \otimes m^\sharp} \quad a^\sharp(f) + (\theta_1 + \theta_2) X_a(f) \\ & \xmapsto{1 \otimes \nu^\sharp \otimes 1} \quad a^\sharp(f) + (\theta_1 + \tilde{\theta}) X_a(f) \\ & \xmapsto{\Delta^\sharp \otimes 1} \quad a^\sharp(f) + \tilde{\theta} X_a(f) + \theta X_a(f) = b^\sharp(f) + \theta X_b(f). \end{aligned}$$

Therefore we have

$$\begin{array}{ccc}
((a, X_a),\ \tilde{\theta}) & \xrightarrow[\Psi_S \times 1]{} & \Big\{(f, df, \theta) \longmapsto (a^\sharp(f), X_a(f),\ \tilde{\theta})\Big\} \\
\Big\downarrow{\scriptstyle \mu_S} & & \Big\downarrow{\scriptstyle T_S} \\
(b, X_b) & \xrightarrow[\Psi_S]{} & (b, X_b) = \Big\{(f, df) \longmapsto (a^\sharp(f) + \tilde{\theta} X_a(f), X_a(f))\Big\},
\end{array}$$

which verifies the commutativity of the above diagram. The lemma is proved.

□

For M and N supermanifolds, define the following "evaluation" map

$$ev : \underline{\mathbf{SM}}(M, N) \times M \to N$$

via its S-points. That means that for any supermanifold S, we need to specify a map ev_S

$$\mathbf{SM}(S \times M, N) \times \mathbf{SM}(S, M) \to \mathbf{SM}(S, N),$$

natural in S. Simply define $ev_S(\varphi, m) = \varphi \circ (1 \times m) \circ \Delta \in \mathbf{SM}(S, N)$, where $\Delta : S \to S \times S$ is the diagonal map.

3.2 Super Lie Groups

Super Lie groups and super Lie algebras. A *super Lie group* G is a supermanifold with maps

$$m : G \times G \to G, \qquad i : G \to G, \qquad \text{and} \qquad e : \mathbf{R}^{0|0} \to G$$

defining the multiplication, inverse and the identity element, such that the usual diagrams expressing the group axioms commute. For example, associativity is given by the commutativity of the diagram:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times 1} & G \times G \\ {\scriptstyle 1 \times m}\downarrow & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{\quad m \quad} & G \end{array}$$

i.e. $m \circ (m \times 1) = m \circ (1 \times m)$.

Yoneda's lemma allows us to interpret a super Lie group G as a representable functor **SM** $\to$ **Group**. We can therefore say that super Lie groups are group objects in **SM**. To determine a group structure on a supermanifold G, one only has to show that the sets $G(S)$, the S-points of G, form groups, in a functorial manner.

Example 1. The easiest example of a super Lie group is $\mathbf{R}^{p|q}$, with group structure:

$$(x, \theta) + (x', \theta') = (x + x', \theta + \theta').$$

This expression is best understood on S-points. An S-point of $\mathbf{R}^{p|q}$ is a collection $(f, g) = (f^1, \ldots, f^p, g^1, \ldots, g^q)$ of p even functions and q odd functions on S (see the comments before Remark 3.1). The group structure on $\mathbf{R}^{p|q}(S)$ is given by:

$$(f, g) + (f', g') = (f + f', g + g').$$

Example 2. $G = GL(p|q)$. Consider first the supermanifold $M_{p|q} \simeq \mathbf{R}^{p^2+q^2|2pq}$ with coordinates written in matrix form:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, D are $p \times p$, respectively $q \times q$ matrices of even elements and B, C are $p \times q$, respectively $q \times p$ matrices of odd elements. ("Even" and "odd" are understood here via S-points.) G is then the open subsupermanifold of $M_{p|q}$ determined by

$GL(p) \times GL(q) \subset |M_{p|q}| = \mathbf{R}^{p^2+q^2}$. An S-point of G is defined by a matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, d (respectively b, c) have entries in $\mathcal{O}_S^0$ = even functions on S (respectively, $\mathcal{O}_S^1$= odd functions on S) and $det(a)det(d)$ is a unit in $\mathcal{O}_S$. With usual matrix multiplication, $G(S)$ is a group. Functoriality in S defines G as a supergroup.

Example 3. Let $\mathbf{R}^{1|1}$ with the following multiplication map $m : \mathbf{R}^{1|1} \times \mathbf{R}^{1|1} \to \mathbf{R}^{1|1}$, defined on S-points by

$$(t, \theta),\ (t', \theta') \overset{m_S}{\longmapsto} (t + t' + \theta\theta', \theta + \theta').$$

Here t and t' are even functions on S, θ and θ' are odd functions on S, and so on... Observe that $\theta\theta'$ is an even function on S. The map m defines a group multiplication on $\mathbf{R}^{1|1}$, with identity given by $(0, 0) \in \mathbf{R}^{1|1}$ and the inverse map given by $(t, \theta) \mapsto (-t, -\theta)$. This is the group structure on $\mathbf{R}^{1|1}$ that we will mostly use in this paper; therefore we are going to call it the *standard* group structure on $\mathbf{R}^{1|1}$.

The super Lie algebra of a super Lie group. As in the classical theory of Lie groups, we can consider left (right) invariant vector fields and identify them with the tangent space at the identity $e \in G$. Let X be a vector field on a super Lie group G, i.e. a graded derivation $X : \mathcal{O}_G \to \mathcal{O}_G$. X is *left-invariant* if the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{m^\sharp} & m_*(\mathcal{O} \otimes \mathcal{O}) \\ \downarrow{\scriptstyle X} & & \downarrow{\scriptstyle 1\otimes X} \\ \mathcal{O} & \xrightarrow[m^\sharp]{} & m_*(\mathcal{O} \otimes \mathcal{O}) \end{array}$$

that is: $m^\sharp \circ X = (1 \otimes X) \circ m^\sharp$. The diagram expresses the fact that X is an infinitesimal right translation.

Consider, for example, $\mathbf{R}^{1|1}$ with the standard group structure defined above. Let Q be the vector field on $\mathbf{R}^{1|1}$ given by $Q = \partial_\theta - \theta\partial_t$, in coordinates (t, θ) on $\mathbf{R}^{1|1}$. Let us show that Q is left-invariant. We need to check that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbf{R}^{1|1}) & \xrightarrow{m^\sharp} & \mathcal{C}^\infty(\mathbf{R}^{1|1}) \otimes \mathcal{C}^\infty(\mathbf{R}^{1|1}) \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle 1\otimes Q} \\ \mathcal{C}^\infty(\mathbf{R}^{1|1}) & \xrightarrow[m^\sharp]{} & \mathcal{C}^\infty(\mathbf{R}^{1|1}) \otimes \mathcal{C}^\infty(\mathbf{R}^{1|1}). \end{array}$$

This is verified by looking at the following two commutative diagrams

$$\begin{array}{ccc} t & \overset{m^\sharp}{\longmapsto} & t_1 + t_2 + \theta_1\theta_2 \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle 1\otimes Q} \\ -\theta & \underset{m^\sharp}{\longmapsto} & -\theta_1 - \theta_2 \end{array} \qquad \begin{array}{ccc} \theta & \overset{m^\sharp}{\longmapsto} & \theta_1 + \theta_2 \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle 1\otimes Q} \\ 1 & \underset{m^\sharp}{\longmapsto} & 1. \end{array}$$

Analogously, a vector field X on a supermanifold M is *right-invariant* if $m^\sharp \circ X = (X \otimes 1) \circ m^\sharp$. One can check for example that the vector field $D = \partial_\theta + \theta\partial_t$ is a right invariant vector field on $\mathbf{R}^{1|1}$. If X, Y are left-invariant vector fields then $[X, Y]$ and $aX + bY$ are also left-invariant vector fields (for a, b real numbers). The set of all left-invariant vector fields with the graded Lie bracket forms a super Lie algebra, i.e. a $\mathbb{Z}/2$-graded vector space endowed with a graded Lie bracket, denoted $\mathfrak{g} = \mathrm{Lie(G)}$. At a point $g \in G$, the vector field X determines a tangent vector $X_g = g^\sharp \circ X$. One can check that $L_{g*}X_e = X_g$, where L_g is left multiplication by g. This justifies the following:

Proposition 3.1. *The map $X \mapsto X_e$, where e is the identity in G, determines an isomorphism of super vector spaces $\alpha : \mathfrak{g} \to T_eG$.*

Proof. α is surjective: Consider v a tangent vector of G at e, and define $X = (1 \otimes v) \circ m^\sharp$. This is a vector field on G with $X_e = v$. Indeed, since $(e^\sharp \otimes 1) \circ m^\sharp = 1$

(by group axioms), we have:

$$v = v \circ \{(e^\sharp \otimes 1) \circ m^\sharp\} = (e^\sharp \otimes v) \circ m^\sharp = e^\sharp \circ (1 \otimes v) \circ m^\sharp = e^\sharp \circ X = X_e.$$

Let's show that X is left invariant. We need to show that:

$$1 \otimes \{(1 \otimes v) \circ m^\sharp\} \circ m^\sharp = m^\sharp \circ (1 \otimes v) \circ m^\sharp.$$

Now, LHS $= (1 \otimes 1 \otimes v) \circ (1 \otimes m^\sharp) \circ m^\sharp$.

RHS$=(m^\sharp \otimes v) \circ m^\sharp = (1 \otimes 1 \circ m^\sharp) \otimes (v \circ 1) \circ m^\sharp = (1 \otimes 1 \otimes v) \circ (m^\sharp \otimes 1) \circ m^\sharp$.

The LHS and RHS are equal since $(1 \otimes m^\sharp) \circ m^\sharp = (m^\sharp \otimes 1) \circ m^\sharp$, again, by one of the group axioms.

α is injective: It is enough to show that a (left-invariant) vector field is determined by its value at e. Indeed, we have:

$$X = (1 \otimes e^\sharp) \circ m^\sharp \circ X = (1 \otimes e^\sharp) \circ (1 \otimes X) \circ m^\sharp = \{1 \otimes (e^\sharp \circ X)\} \circ m^\sharp = (1 \otimes X_e) \circ m^\sharp$$

where in the second equality we used that X is left-invariant. □

Proposition 3.2. *The isomorphism α induces a super Lie algebra structure on T_eG, with Lie bracket:*

$$[v, w] = (v \otimes w - w \otimes v) \circ m^\sharp$$

for v, w tangent vectors at e.

Proof. Let X and Y be the (left-invariant) vector fields determined by v, respectively w. The induced Lie bracket on T_eG is:

$$[v, w] = e^\sharp \circ [X, Y].$$

To prove the proposition we need to show that:

$$1 \otimes [(v \otimes w - w \otimes v) \circ m^\sharp] \circ m^\sharp = [X, Y].$$

We have

$$\begin{aligned} LHS &= (1 \otimes v \otimes w - 1 \otimes w \otimes v) \circ (1 \otimes m^\sharp) \circ m^\sharp. \\ RHS &= [1 \otimes v \circ m^\sharp, 1 \otimes w \circ m^\sharp] \\ &= 1 \otimes v \circ m^\sharp \circ 1 \otimes w \circ m^\sharp - 1 \otimes w \circ m^\sharp \circ 1 \otimes v \circ m^\sharp \\ &= (1 \otimes v \otimes w - 1 \otimes w \otimes v) \circ (m^\sharp \otimes 1) \circ m^\sharp. \end{aligned}$$

The two terms are the same. □

Remark 3.3. *If instead of left-invariant vector fields we consider right-invariant vector fields, the corresponding Lie algebra* $\mathfrak{g}^{op}$ *will induce a Lie bracket* $[\ ,\ ]_r$ *on* T_eG *which is the opposite of the left Lie bracket* $[\ ,\ ]$. *This happens because the inverse map* $i : g \mapsto g^{-1}$ *of* G *takes left invariant vector fields to right invariant, and on the tangent space at the identity the map* i *induces the isomorphism* $x \mapsto -x$.

Remark 3.4. *As in the classical case, the tangent sheaf* $\mathcal{T}G$ *is a* globally *free* $\mathcal{O}_G$*-module of rank the dimension of* G*, with a basis consisting of a basis for left invariant vector fields, i.e:*

$$\mathcal{O}_G \otimes \mathfrak{g} \cong \mathcal{T}G.$$

Group Actions. We will mostly talk about right actions since we have in mind principal bundles. Left actions are defined similarly. Let P be a supermanifold. A *right action* of a super Lie group G on P is a map $\mu : P \times G \to G$ such that $\mu \circ (1 \times e) = 1$ and the following diagram commutes:

$$\begin{array}{ccc} P \times G \times G & \xrightarrow{\mu \times 1} & P \times G \\ \Big\downarrow{\scriptstyle 1 \times m} & & \Big\downarrow{\scriptstyle \mu} \\ P \times G & \xrightarrow[\mu]{} & P. \end{array}$$

Looking at $\mathbf{R}^{0|0}$-points, one sees the usual action of the group G_{red} on the manifold M_{red}.

Examples: The right action of a supergroup G on itself. The action of G on the product $M \times G$, with action on the second factor by right multiplication. The (left) action of $GL(p|q)$ on $\mathbf{R}^{p|q}$ by usual matrix multiplication. (Again, this is best understood on S-points.)

A map of G-manifolds $f : P \to Q$ is *G-equivariant* if the following diagram

$$\begin{array}{ccc} P \times G & \xrightarrow{f \times 1} & Q \times G \\ \mu \downarrow & & \downarrow \nu \\ P & \xrightarrow[f]{} & Q \end{array}$$

commutes.

We can also define a right action of G on a *relative supermanifold* $p : P \to M$ is a map $\mu : P \times G \to P$ such that:

$$\begin{array}{ccc} P \times G & \xrightarrow{\mu} & P \\ p_1 \downarrow & & \downarrow p \\ P & \xrightarrow[p]{} & M. \end{array}$$

commutes, where p_1 is projection onto the first factor. If $P \xrightarrow{p} M$ and $Q \xrightarrow{q} M$ are relative supermanifolds over M, the map $f : P \to Q$ is *G-equivariant over M* if it is G-equivariant and if $q \circ f = p$.

Consider $V \subseteq P$ invariant under G, i.e. $|V|$ is invariant under $|G|$. We can interpret sections of the structure sheaf over $|V|$ as:

$$\mathcal{O}_P(|V|) = \{f : V \to \mathbf{R}^{1|1}\}$$

We can define:

$$\mathcal{O}_P(|V|)^G = \{f : V \to \mathbf{R}^{1|1} | f \text{ is G-equivariant}\}$$

where the action of G on $\mathbf{R}^{1|1}$ is trivial. In this way we can give meaning to the notion of *invariant functions* on a supermanifold.

As for algebraic groups, the *quotient* of an action μ of G on a supermanifold P is defined as the pushout of the diagram:

$$\begin{array}{ccc} P \times G & \xrightarrow{\mu} & P \\ {\scriptstyle p_1}\downarrow & & \\ P & & \end{array}$$

Explicitly, the quotient is a relative supermanifold $p : P \to M$ acted on by G, which is universal, in the sense that, given a relative supermanifold $f : P \to N$ acted by G, there is a unique map $g : M \to N$ such that $f = g \circ p$. Of course, the quotient does not always exist in this sense.

Proposition 3.5. *Let $p : P \to M$ be the quotient of the action of G on P. Then:*

$$\mathcal{O}_M \xrightarrow{\simeq} (p_* \mathcal{O}_P)^G.$$

Therefore, $\mathcal{O}_M$ fits into the exact sequence:

$$0 \to \mathcal{O}_M \xrightarrow{p^\sharp} p_* \mathcal{O}_P \xrightarrow{\mu^\sharp - p_1^\sharp} \pi_* (\mathcal{O}_P \otimes \mathcal{O}_G)$$

where $\pi = p \circ \mu = p \circ p_1$.

Proof. For any open $U \subseteq M$ open, we must have:

$$\mathcal{O}_M(|U|) \xrightarrow{\simeq} (p_* \mathcal{O}_P)^G (p^{-1}(|U|)).$$

This is obtained from the definition of quotients, by considering $N = \mathbf{R}^{1|1}$. □

G-invariant vector fields. Let P be a supermanifold, with G-action μ. A (graded) vector field is *G-invariant* if the diagram:

$$\begin{array}{ccc} \mathcal{O}_P & \xrightarrow{\mu^\sharp} & \mu_*(\mathcal{O}_P \otimes \mathcal{O}_G) \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle X\otimes 1} \\ \mathcal{O}_P & \xrightarrow[\mu^\sharp]{} & \mu_*(\mathcal{O}_P \otimes \mathcal{O}_G) \end{array}$$

is commutative. As before, one can check $R_{g*}X_p = X_{pg}$, for $p \in P$ and $g \in G$. To any $X \in \mathfrak{g}$ we can associate a *fundamental vector field* X^* defined by:

$$X^* = (1 \otimes X) \circ \mu^\sharp.$$

Fundamental vector fields are not invariant. Instead, we have:

$$R_{g*}X^* = (ad(g^{-1})X)^*$$

where $\mathrm{ad}(x) = (L_x \circ R_{x^{-1}})_* : \mathfrak{g} \to \mathfrak{g}$ is the usual adjoint map. As in the proof of Prop. 3.1, one can show:

Proposition 3.6. $[X^*, Y^*] = [X, Y]^*$

therefore, the map:

$$\mathfrak{g} \to \mathcal{T}P(|P|) : X \to X^*$$

is a morphism of super Lie algebras.

E.g. Fundamental vector fields for the right action of G on itself are exactly the left invariant vector fields on G. On $M \times G$ with the action of G by right multiplication on the second factor, the fundamental vector fields are of the form: $X^* = 1 \otimes [(1 \otimes X) \circ m^\sharp]$.

3.3 Super Vector Bundles

A *super vector bundle* of rank $p|q$ over a supermanifold M can be viewed in two ways:

· A locally free $\mathcal{O}_M$-module $\mathcal{E}$ of dimension $p|q$.

· A relative supermanifold $E \to M$, that is locally a product over M, with typical fiber $\mathbf{R}^{p|q}$ and structure group $GL(p|q)$.

Given a locally free $\mathcal{O}_M$-module $\mathcal{E}$ of rank $p|q$ over M, we can define the supermanifold E by its functor of points. Namely, consider the functor:

$$S \mapsto \{(f, s) | f : S \to M, \text{and } s \text{ is an even section of } f^*\mathcal{E}.\}$$

This is a contravariant functor. To see that is representable, it is enough to consider the question locally on M. Then $\mathcal{E}$ is free with basis, let us say, $\{e_1, \ldots, e_p, f_1, \ldots, f_q\}$. Then the functor is represented by $M \times \mathbf{R}^{p|q}$. If we change the trivialization of $\mathcal{E}$ (i.e. we choose a different basis for $\mathcal{E}$), we get a map between the two representable functors

$$\mathbf{SM}(\,\cdot\,, M \times \mathbf{R}^{p|q}) \to \mathbf{SM}(\,\cdot\,, M \times \mathbf{R}^{p|q}),$$

which is induced by a map

$$M \times \mathbf{R}^{p|q} \to M \times \mathbf{R}^{p|q} : 1_M \times \{M \to GL(p|q)\}.$$

This map makes E a bundle with structure group $GL(p|q)$.

E.g. The *trivial* super bundle $p : M \times \mathbf{R}^{p|q} \to M$. If $E \to M$ is a super bundle, and U is open in M, then $E|_U$ is also a super bundle. We've already seen that the sheaf of derivations $\mathcal{T}M$ is a locally free $\mathcal{O}_M$-module, giving thus the *tangent bundle* $TM \to M$ of M.

Remark 3.1. *Note that in the light of the first definition of a super vector bundle, a super vector bundle over an ordinary manifold M is just a $\mathbb{Z}/2$-graded vector bundle over M.*

3.4 Differential Equations on Supermanifolds

Let M be a supermanifold of dimension (p, q) and let X be an odd vector field on M. An *integral curve* for X is a curve $c : \mathbf{R}^{1|1} \to M$ such that

$$D \circ c^\sharp = c^\sharp \circ X, \tag{3.1}$$

i.e. a solution to the differential equation (3.1) defined by X. Here $D = \partial_\theta + \theta\partial_t$ is the standard right invariant vector field on $\mathbf{R}^{1|1}$. The existence and uniqueness of solutions of a differential equation is a local problem and reduces to the classical problem.

To see this, let X be an odd vector field on $\mathbf{R}^{p|q}$. Then X can be written

$$X = \sum a_i \frac{\partial}{\partial x^i} + \sum b_j \frac{\partial}{\partial \theta^j},$$

with a_i and b_j odd, respectively even, function on $\mathbf{R}^{p|q}$, $i = 1, \ldots, p$, $j = 1, \ldots, q$. In coordinates, the curve c can be written

$$\begin{cases} x^i = x^i(t), & i = 1, \ldots, p \\ \theta^j = \theta^j(t, \theta) = y^j(t)\theta, & j = 1, \ldots, q. \end{cases}$$

If we rewrite (3.1) as $\partial_Q c = X \circ c$, this gives rise to the system

$$\begin{cases} \theta \dot{x}^i(t) = a_i(x^1(t), \ldots, x^p(t), y^1(t)\theta, \ldots, y^q(t)\theta) \\ y^j(t) = b_j(x^1(t), \ldots, x^p(t), y^1(t)\theta, \ldots, y^q(t)\theta) = b_j(x^1(t), \ldots, x^p(t)) \end{cases}$$

where the last equality is true since the b's are even. Therefore the system reduces to a first order system of p differential equations, which has a unique solution, by the general theory of differential equations, once we fix an initial condition. We conclude that (3.1) admits a solution which is unique once we know the bosonic value of the curve for a fixed time t_0.

We next want to show continuity on the initial data of the solutions of (3.1). For that, we need a family version of the equation (3.1). Let us begin with the following lemma.

Lemma 3.1. *Let X be an even vector field on a compact supermanifold M (i.e. the underlying manifold is compact). Then there exists a unique map $c : \mathbf{R} \times M \to M$ satisfying the following two conditions:*

$$\begin{cases} \partial_t \circ c^\sharp = c^\sharp \circ X \\ c\big|_{0\times M} = id_M. \end{cases}$$

The map c is called the *flow of the vector field X*.

Proof. The existence and uniqueness of a global solution follows from the existence and uniqueness of a local solution, since M is compact. To solve the local problem, we can w.l.o.g. assume that $M = \mathbf{R}^{p|q}$. Let $x^1, \dots, x^{p+q}$ be the coordinate functions on $\mathbf{R}^{p|q}$, with the first p coordinates even, and the last q odd. We also write $\theta_1, \dots, \theta_q$ for the last q odd coordinates. Let c^i be the image of x^i under the map $c^\sharp$. Let's write

$$c^i = \sum c^i_J \theta^J, \text{ with } c^i_J \in \mathcal{C}^\infty(\mathbf{R} \times \mathbf{R}^p).$$

The vector field X can be written $X = \sum_1^{p+q} a_i \partial_{x^i}$, with a_i even, $i = 1, \dots, p$, respectively odd, $i = p+1, \dots p+q$, functions on $\mathbf{R}^{p|q}$. We further write

$$a_i = \sum a^i_J \theta^J, \text{ with } a^i_J \in \mathcal{C}^\infty(\mathbf{R}^p),$$

with some of the a^i_J possibly zero. The relation (1) holds for a map $c : I \times M \to M$, with I a small neighborhood of 0, if and only if it holds when evaluated on the coordinate functions x^i on $\mathbf{R}^{p|q}$. Consequently, we must have that

$$\partial_t c^i = c^\sharp(a_i).$$

Equivalently, we have

$$\begin{aligned}
\sum \frac{dc^i_J}{dt}(t,x)\theta^J &= a_i(c(t,x,\theta)) \\
&= a_i(\sum_J c_J(t,x)\theta^J) \\
&= a_i(c_0(t,x) + \sum_{J\neq 0} c_J(t,x)\theta^J) \\
&= a_i(c_0(t,x)) + \sum \frac{\partial a_i}{\partial x^j}(c_0(t,x))c^j_J(t,x)\theta^J + \ldots \\
&= a_i(c_0(t,x)) + \sum f^i_J(\frac{\partial^L a_i}{\partial x^L}(c_0(t,x)), c_K(t,x))\theta^J,
\end{aligned}$$

where f^i_J are polynomial functions on some large euclidean space, $|L| \leq p$, and $|K| \leq q$. The fourth equality comes from the Taylor expansion for the function a_i around $c_0(t,x)$. Equating the coefficients of the above relation, we obtain the system

$$\begin{cases}
\frac{dc^i_0}{dt}(t,x) = a_i(c_0(t,x)), \quad i = 1,\ldots,p \\
\frac{dc^i_J}{dt}(t,x) = \sum f^i_J(\frac{\partial^L a_i}{\partial x^L}(c_0(t,x)), c_K(t,x)), \quad 0 \neq |J| \leq q
\end{cases}$$

We solve first the system of the first p equations to determine a_0, and then the first order system of differential equations determined by the last $(p+q)(2^q-1)$ equations. The initial condition of the system is given by the relations

$$x^i = \sum c^i_J(0,x)\theta^J, \quad i = 1,\ldots,p+q,$$

which reflect the condition (2) in the statement of the lemma. By the general theory of systems of differential equations, the above system admits a unique solution. The lemma follows. □

More generally, given an even vector field X on a supermanifold M, and a parametrizing supermanifold S, we have a unique solution $\alpha : \mathbf{R} \times S \to M$ of the system

$$\begin{cases} \partial_t \circ \alpha^\sharp = \alpha^\sharp \circ X \\ \alpha\big|_{0\times S} = f \end{cases}$$

for some initial condition $f : S \to M$. It is given by

$$\alpha = c \circ (1 \times f)$$

where c is the flow determined by X.

Let's come back and show the continuity of the flow of an odd vector field on a supermanifold.

Lemma 3.2. *Let M be a compact supermanifold and X be an odd vector field on M. Then there exists a unique map $\alpha : \mathbf{R}^{1|1} \times S \to M$ satisfying the following two conditions:*

$$\begin{cases} D \circ \alpha^\sharp = \alpha^\sharp \circ X \\ \alpha\big|_{0\times S} = f \end{cases}$$

for some initial condition $f : S \to M$. Here $D = \partial_\theta + \theta\partial_t$ is as usual.

Proof. Again, it is enough to solve the problem locally, for which we can assume that $M = \mathbf{R}^{p|q}$. Write $X = \sum a_i \partial_{x^i}$. Then the first relation on arbitrary functions g on $\mathbf{R}^{p|q}$ gives

$$\Big(\frac{\partial g}{\partial x^i} \circ \alpha\Big)\frac{\partial \alpha^i}{\partial \theta} + \theta\Big(\frac{\partial g}{\partial x^i} \circ \alpha\Big)\frac{\partial \alpha^i}{\partial t} = \Big(a_i \frac{\partial g}{\partial x^i}\Big) \circ \alpha. \tag{3.2}$$

Let us write $\alpha = G + \theta H$, with $G, H \in \mathcal{C}^\infty(I \times S)$, for some I a neighborhood of 0. Then, by Taylor's expansion, we have

$$a_i(\alpha) = a_i(G) + \theta \frac{\partial a_i}{\partial x^j}(G) H^j$$

and (3.2) becomes

$$H^i + \theta\Big(\frac{\partial G^i}{\partial t} + \theta \frac{\partial H^i}{\partial t}\Big) = a_i(G) + \theta \frac{\partial a_i}{\partial x^j}(G) H^j.$$

This is equivalent to the system

$$\begin{cases} a_i(G) = H^i \\ \frac{\partial G^i}{\partial t}(s,t) = \frac{\partial a_i}{\partial x^j}(G(s,t)) H^j(s,t) \end{cases}$$

which gives rise to the system

$$\frac{\partial G^i}{\partial t}(s,t) = \frac{\partial a_i}{\partial x^j}(G(s,t)) a_j(G(s,t)). \tag{3.3}$$

Now, $\frac{\partial a}{\partial x^j} a_j$ is an *even* vector field on $\mathbf{R}^{p|q}$, so, by the previous lemma, the system (3.3) admits a unique solution once we know $G(0,s)$, which is given by the initial condition $f : S \to M$. The lemma is proved. $\square$

Let X be an odd vector field on a supermanifold M, and let $\alpha : M \times \mathbf{R}^{1|1} \to M$ be the flow of X. By definition, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{C}^\infty(M) & \xrightarrow{\alpha^\sharp} & \mathcal{C}^\infty(\mathbf{R}^{1|1} \times M) \\ \downarrow X & & \downarrow Q \\ \mathcal{C}^\infty(M) & \xrightarrow{\alpha^\sharp} & \mathcal{C}^\infty(\mathbf{R}^{1|1} \times M). \end{array}$$

Let $u : S \to \mathbf{R}^{1|1} \times M$ be an S-point of $\mathbf{R}^{1|1} \times M$. Then

$$u^\sharp \circ Q \circ \alpha^\sharp = u^\sharp \circ \alpha^\sharp \circ X,$$

which is to say that

$$\alpha_{*u}(Q_u) = X_{\alpha(u)},$$

where α_* is the differential of α. If we denote $u = (t, \theta, x)$, then the equation above can also be written

$$\partial_Q \alpha(t, \theta, x) = X(\alpha(t, \theta, x)).$$

This relation probably justifies our way of looking at a differential equation as a commutative diagram. (See also [Šan80].)

Again, let X be an odd vector field on a supermanifold M. By the lemma above, X defines a flow $\alpha : \mathbf{R}^{1|1} \times M \to M$. Define the map $\alpha_0 : \mathbf{R} \times M \to M$ by $\alpha_0 = \alpha \circ (i \times 1_M)$, where $i : \mathbf{R} \to \mathbf{R}^{1|1}$ is the standard inclusion map. Moreover i is a group homomorphism, if $\mathbf{R}$ and $\mathbf{R}^{1|1}$ are endowed with the standard group structures (see the previous section). Therefore α_0 defines a flow map.

Lemma 3.3. *The map α_0 is the flow of the even vector field X^2.*

Proof. Indeed, by definition $\alpha^\sharp \circ X = D \circ \alpha^\sharp$. Therefore

$$\begin{aligned} \alpha^\sharp \circ X^2 &= D \circ \alpha^\sharp \circ X \\ &= D \circ D \circ \alpha^\sharp \\ &= \partial_t \circ \alpha^\sharp. \end{aligned}$$

Since ∂_t commutes with $i^\sharp \otimes 1$, the claim follows. □

Example: Let D be the usual vector field on $\mathbf{R}^{1|1}$. Then the flow of D is given by the group multiplication map $m : \mathbf{R}^{1|1} \times \mathbf{R}^{1|1} \to \mathbf{R}^{1|1}$. To see this, we should verify that m fits into the diagram

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbf{R}^{1|1}) & \xrightarrow{m^\sharp} & \mathcal{C}^\infty(\mathbf{R}^{1|1} \times \mathbf{R}^{1|1}) \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D\otimes 1} \\ \mathcal{C}^\infty(\mathbf{R}^{1|1}) & \xrightarrow{m^\sharp} & \mathcal{C}^\infty(\mathbf{R}^{1|1} \times \mathbf{R}^{1|1}). \end{array}$$

This is indeed the case: the diagram expresses the fact D is a right invariant vector field.

Remark 3.4. *We could have written the flow of an odd vector field by differentiating the supercurve* $c : M \times \mathbf{R}^{1|1} \to M$ *in the direction of the vector field* $Q = \partial_\theta - \theta\partial_t$ *on* $\mathbf{R}^{1|1}$*. To distinguish between the two flows, one could call them* D*-flows and* Q*-flows, but hopefully it will be clear from context what type of flow we are using.*

CHAPTER 4

THE SUPER PARALLEL TRANSPORT OF CONNECTIONS

The purpose of this section is to describe the super parallel transport of a connection on a super vector bundle over a supermanifold. This follows closely the geometric idea of parallel transport associated to a connection on a vector bundle over a manifold.

4.1 Setup

Let E be a super vector bundle over a supermanifold M, and let ∇ be a connection on E (see [DM99]), i.e. $\nabla : \Gamma(M, E) \to \Omega^1(M, E)$ such that

$$\nabla(fs) = df \otimes s + f\nabla s, \qquad f \in \mathcal{C}^\infty(M), \quad s \in \Gamma(M, E).$$

In particular, for $X \in \mathcal{X}(M)$ a vector field on M, we have $\nabla_X : \Gamma(M, E) \to \Gamma(M, E)$ with

$$\nabla_X(fs) = X(f)s + (-1)^{p(X)p(f)} f\nabla_X s.$$

Let $c : S \times \mathbf{R}^{1|1} \to M$ be a (family of) supercurve(s parametrized by a supermanifold S) in M. Consider the pull-back connection $c^*\nabla$ and the derivation $(c^*\nabla)_D : \Gamma(c^*E) \to \Gamma(c^*E)$. Here D is the vector field $\partial_\theta + \theta\partial_t$ on $\mathbf{R}^{1|1}$, extended trivially to $S \times \mathbf{R}^{1|1}$. An element of $\Gamma(c^*E)$ is called a *section of E along c*. We say that the section s along c is *super-parallel* if

$$(c^*\nabla)_D s = 0.$$

In local coordinates, we can think of this as being a *half-order differential equation*. There are two reasons for that: first, the vector field D squares to the vector field $\frac{d}{dt}$, second, for $2n$ unknown functions we need as initial data n values.

Proposition 4.1. *Let $c : S \times \mathbf{R}^{1|1} \to M$ be a supercurve in the compact supermanifold M (i.e. the reduced manifold is compact). Let $\psi_0 \in \Gamma(c_{0,0}^*E)$ be a section of E along $c_{0,0} : S \to S \times \mathbf{R}^{1|1} \to M$, with the first map the standard inclusion $i_{0,0} : S \to S \times \mathbf{R}^{1|1}$. Then, there exists a unique super-parallel section ψ of E along c, such that $\psi(0,0) = \psi_0$.*

Proof. The fact that ψ extends to all of $S \times \mathbf{R}^{1|1}$ is a standard argument on the flows of vector fields on *compact* manifolds. The existence (and uniqueness) of ψ is then a local problem. Let $U \subseteq M$ be a trivializing neighborhood such that $E_{|U} \cong U \times \mathbf{R}^{p|q}$ ($p|q$ is the rank of the bundle E). Then the connection can be written as $\nabla = d + A$, for some $A \in \Omega^1(M) \otimes End(\mathbf{R}^{p|q})^{ev}$. The equation $(c^*\nabla)_D s = 0$, with the given initial condition is then equivalent to the system

$$\begin{cases} \frac{\partial \psi}{\partial D}(s,t,\theta) + A(s,t,\theta)\psi(s,t,\theta) = 0 \\ \psi(s,0,0) = \psi_0(s) \end{cases}$$

where ψ is defined in a neighborhood of $S \hookrightarrow S \times \mathbf{R}^{1|1}$ with values in $\mathbf{R}^{p|q}$, and $A : S \times \mathbf{R}^{1|1} \to End(\mathbf{R}^{p|q})$ is short for $(c^*A)(D)$. If we write

$\psi(s,t,\theta) = (a^i(s,t) + \theta b^i(s,t))_{i=1,\dots,p+q}$

$A(s,t,\theta) = (c^{ij}(s,t) + \theta d^{ij}(s,t))_{i,j=1,\dots,p+q}$

then the system is equivalent to

$$\begin{cases} b^i(s,t) = -c^{ij}(s,t)a^j(s,t) \\ \frac{da^i}{dt}(s,t) = -\varepsilon(c^{ij}(s,t))b^j(s,t) - d^{ij}(s,t)a^j(s,t) \\ a^i(s,0) = \psi_0^i(s) \end{cases}$$

Here $\varepsilon(a) = \begin{cases} a, & \text{if } a \text{ is even} \\ -a, & \text{if } a \text{ is odd} \end{cases}$

It is clear that this system admits a unique solution around $S \times (0,0)$. The proposition is proved. □

Lemma 4.2. *Let $c : S \times \mathbf{R}^{1|1} \to M$ be a supercurve in M, and let $\varphi : S' \to S$ be an arbitrary map. Consider the supercurve $c' : S' \times \mathbf{R}^{1|1} \to M$ defined by $c' = c \circ \bar{\varphi}$, where $\bar{\varphi} = \varphi \times 1_{\mathbf{R}^{1|1}}$. If ψ is a parallel section along c, then $\psi \circ \bar{\varphi}$ is parallel along $c \circ \bar{\varphi}$.*

Proof. ψ is parallel along c if $(c^*\nabla)_D\psi = 0$. Let us observe that $\bar{\varphi}_* D = D$. Indeed, an easy check shows that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{C}^\infty(S) \otimes \mathcal{C}^\infty(\mathbf{R}^{1|1}) & \xrightarrow{\varphi^\sharp \otimes 1} & \mathcal{C}^\infty(S') \otimes \mathcal{C}^\infty(\mathbf{R}^{1|1}) \\ \downarrow{\scriptstyle 1\otimes D} & & \downarrow{\scriptstyle 1\otimes D} \\ \mathcal{C}^\infty(S) \otimes \mathcal{C}^\infty(\mathbf{R}^{1|1}) & \xrightarrow[\varphi^\sharp \otimes 1]{} & \mathcal{C}^\infty(S') \otimes \mathcal{C}^\infty(\mathbf{R}^{1|1}). \end{array}$$

We therefore have

$$\begin{aligned} (\bar{\varphi}^* c^* \nabla)_D(\bar{\varphi}^*\psi) &= \bar{\varphi}^\sharp\big((c^*\nabla)_{\varphi_* D}\psi\big) \\ &= \bar{\varphi}^\sharp\big((c^*\nabla)_D\psi\big) \\ &= 0, \end{aligned}$$

since ψ is parallel. That is, $\psi\bar{\varphi}$ is parallel along $c\bar{\varphi}$.

□

We interpret this Lemma by saying that the parallel transport is *natural in S*.

Remark 4.3. *We could have as well defined a super-parallel section along a super-curve c in M to be a section s along c that satisfies the equation*

$$(c^*\nabla)_Q s = 0,$$

where $Q = \partial_\theta - \theta\partial_t$. Let us call such sections Q-parallel to distinguish them from the parallel sections defined above. Their relevance will be conspicuous in Property (3) of Section 4.6.

4.2 Invariance under reparametrization

The usual parallel transport is invariant under reparametrization. We will see in this subsection what that means in the super-context.

Lemma 4.1. *Let $\varphi : \mathbf{R}^{1|1} \to \mathbf{R}^{1|1}$ be a diffeomorphism of $\mathbf{R}^{1|1}$ that preserves the distribution $\mathcal{D}$ determined by the standard vector field D on $\mathbf{R}^{1|1}$. If we write $\varphi(t,\theta) = (a(t), b(t)\theta)$ then we have $b^2 = \frac{da}{dt}$.*

Proof. We require that φ satisfies $\varphi_* D = c \cdot D$, for some even function c on $\mathbf{R}^{1|1}$. Now $\varphi_* D$ fits into the diagram

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbf{R}^{1|1}) & \xrightarrow{\varphi^\sharp} & \mathcal{C}^\infty(\mathbf{R}^{1|1}) \\ {\scriptstyle \varphi_* D}\downarrow & & \downarrow{\scriptstyle D} \\ \mathcal{C}^\infty(\mathbf{R}^{1|1}) & \xrightarrow{\varphi^\sharp} & \mathcal{C}^\infty(\mathbf{R}^{1|1}). \end{array}$$

Following the diagram both ways we have

$$\begin{array}{ccc} t & \xmapsto{\varphi^\sharp} & a(t) \\ {\scriptstyle \varphi_* D}\downarrow & & \downarrow{\scriptstyle D} \\ \theta c & \xmapsto{\varphi^\sharp} & \theta b(t)c(a(t)) = \theta\frac{da}{dt} \end{array} \qquad \begin{array}{ccc} \theta & \xmapsto{\varphi^\sharp} & b(t)\theta \\ {\scriptstyle \varphi_* D}\downarrow & & \downarrow{\scriptstyle D} \\ c & \xmapsto{\varphi^\sharp} & c(a(t)) = b(t). \end{array}$$

The conclusion follows. □

Let $c : S \times \mathbf{R}^{1|1} \to M$ be a supercurve in M and let ψ be a parallel section of E along c, i.e. $(c^*\nabla)_D\psi = 0$. Let φ be a family of diffeomorphisms of $\mathbf{R}^{1|1}$ that preserve the distribution $\mathcal{D}$, parametrized by S. Then we have

$$((c \circ \varphi)^*\nabla)_D(\psi \circ \varphi) = ((c^*\nabla)_{\varphi_* D}\psi) \circ \varphi = b \cdot ((c^*\nabla)_D\psi) \circ \varphi = 0.$$

Therefore, we conclude that if ψ is a parallel section of E along c, then $\psi \circ \varphi$ is a parallel section of E along $c \circ \varphi$. We say that the parallel transport defined by the connection is *invariant under reparametrization*. (In our case, "reparametrization" refers to diffeomorphisms that preserve a distribution.)

4.3 Compatibility with the "old" parallel transport

Let $c : \mathbf{R}^{1|1} \to M$ be a superpath in M that maps to the reduced part of M, i.e. c factors as a map $\mathbf{R}^{1|1} \to \mathbf{R} \to M$, where the first map is the standard projection $p : \mathbf{R}^{1|1} \to \mathbf{R}$. Let also $\varphi : \mathbf{R}^{1|1} \to \mathbf{R}^{1|1}$ be a diffeomorphism of $\mathbf{R}^{1|1}$ that preserves the distribution $\mathcal{D}$. Write $\varphi(t, \theta) = (s, \eta) = (a(t), b(t)\theta)$. We could also write $\varphi = a + b\theta$ by interpreting the diffeomorphism φ as a function on $\mathbf{R}^{1|1}$ with a, $b\theta$ the even, respectively the odd part of φ. On the reduced manifolds the map φ is just a. We will see in Lemma 6.1 that

$$(c^*\nabla)_D = \partial_\eta + \eta(|c|^*\nabla)_{\partial_s}.$$

$D = \partial_\eta + \eta\partial_s$ denotes the standard vector field on $\mathbf{R}^{1|1}$ in coordinates (s, η). In particular, this relation tells us that a section $\psi \in \Gamma(|c|^*E)$ is parallel along $|c|$ with respect to ∇ if and only if ψ viewed as a section of c^*E is super-parallel along c with respect to ∇. We also have

$$((c\varphi)^*\nabla)_{\tilde{D}} = \partial_\theta + \theta(|c\varphi|^*\nabla)_{\partial_t}.$$

Here $\tilde{D} = \partial_\theta + \theta\partial_t$ is the standard vector field on $\mathbf{R}^{1|1}$ in coordinates (t, θ). (We use different letters for the same object to avoid confusions when applying the chain rule.) Let $\psi = \psi_1 + \eta\psi_2$ be a section of E along c. Then

$$\psi \circ \varphi = (\psi_1 + \eta\psi_2)(a(t), \theta b(t)) = \psi_1(a(t)) + \theta b(t)\psi_2(a(t)).$$

As before, we have

$$((c\varphi)^*\nabla)_{\tilde{D}}(\psi \circ \varphi) = b \cdot ((c^*\nabla)_D\psi) \circ \varphi.$$

On one side,

$$\begin{aligned} LHS &= \partial_\theta(\psi_1(a(t)) + \theta b(t)\psi_2(a(t))) + \theta(|c\varphi|^*\nabla)_{\partial_t}\psi_1(a(t)) \\ &= b(t)\psi_2(a(t)) + \theta(|c\varphi|^*\nabla)_{\partial_t}\psi_1(a(t)). \end{aligned}$$

On the other side,

$$RHS = b(t)\psi_2(a(t)) + \theta b(t) \cdot b(t)((|c|^*\nabla)_{\partial_s}\psi_1 \circ a)(t).$$

Comparing the two sides and taking into account that $b^2 = a'$, we obtain

$$(a^*|c|^*\nabla)_{\partial_t}(\psi_1 \circ a) = a'(t) \cdot ((|c|^*\nabla)_{\partial_s}\psi_1) \circ a,$$

which expresses the invariance under reparametrization of the usual notion of parallel transport of connections. In this way, we generalized the old concept of parallel transport along paths in M, to a new concept of "super parallel transport" along superpaths in M in the sense that a parallel section in the old sense is super-parallel in the new sense, and the new parametrization invariance is compatible as above with the parametrization invariance in the old sense.

4.4 Recovering the connection from the super parallel transport

The next topic we want to address is the following: Given a connection on a superbundle and its associated parallel transport, how can we recover the connection? The answer goes as follows.

To give a connection ∇ on E over M amounts to specifying for each vector field X on M an X-derivation $\nabla_X = \tilde{X} : \Gamma(M, E) \to \Gamma(M, E)$, i.e.

$$\tilde{X}(fs) = X(f)s + (-1)^{p(X)p(f)} f\tilde{X}(s), \quad f \in \mathcal{C}^\infty(M), \quad s \in \Gamma(M, E),$$

such that the correspondence $X \mapsto \tilde{X}$ is $\mathcal{C}^\infty(M)$-linear.

Let X be an *odd* vector field on M, and let $\alpha = \alpha_X : \mathbf{R}^{1|1} \times M \to M$ be the flow of X. By definition (see Section 3.4), X fits into the following diagram

$$\begin{array}{ccc} \mathcal{C}^\infty(M) & \xrightarrow{\alpha^\sharp} & \mathcal{C}^\infty(\mathbf{R}^{1|1} \times M) \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle D} \\ \mathcal{C}^\infty(M) & \xrightarrow{\alpha^\sharp} & \mathcal{C}^\infty(\mathbf{R}^{1|1} \times M). \end{array}$$

The pullback-connection via the path α will define, via the vector field D, a D-derivation $\tilde{D}$, on the sections of the pull-back bundle

$$\Gamma(\mathbf{R}^{1|1} \times M, \alpha^* E) = \mathcal{C}^\infty(\mathbf{R}^{1|1} \times M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(M, E).$$

Lemma 4.1.

$$\tilde{D} = D \otimes 1 + 1 \otimes \tilde{X},$$

where the right hand side is defined by

$$f \otimes s \mapsto Df \otimes s + f \otimes \tilde{X}s, \quad f \in \mathcal{C}^\infty(\mathbf{R}^{1|1} \times M), \quad s \in \Gamma(M, E).$$

Proof. Indeed, both sides are D-derivations, and they coincide on sections of E pulled-back via the map α. To see the latter, let us write $\alpha^* s = 1 \otimes s$, for $s \in \Gamma(M, E)$. Then

$$\tilde{D}(1 \otimes s) = (\alpha^* \nabla)_D(\alpha^* s) = \alpha^*(\nabla_X s) = 1 \otimes (\tilde{X} s) = (D \otimes 1 + 1 \otimes \tilde{X})(1 \otimes s).$$

□

Now, the parallel transport depicts in particular the parallel sections along c in the direction of D. That information is enough to determine $\tilde{D} : \Gamma(c^* E) \to \Gamma(c^* E)$ as a D-derivation. Indeed, locally, if s_i, $i = 1, \ldots, p+q$ are linearly independent parallel sections, then any $s \in \Gamma(c^* E)$ can be written $s = \sum f_i s_i$, with $f_i \in \mathcal{C}^\infty(\mathbf{R}^{1|1} \times M)$. Then, $\tilde{D}(\sum f_i s_i) = \sum D(f_i) s_i$. By the Lemma above, we have in particular $\tilde{D}(\alpha^* s) = \alpha^*(\tilde{X} s)$, for $s \in \Gamma(M, E)$, and since $\alpha^* : \Gamma(M, E) \to \Gamma(\mathbf{R}^{1|1} \times M, \alpha^* E)$ is injective, knowing $\tilde{D}$, uniquely determines $\tilde{X}$.

Let now X be an *even* vector field on M and let $\alpha : \mathbf{R} \times M \to M$ be the flow determined by X. Let $\hat{\alpha} : \mathbf{R}^{1|1} \times M \to M$ be the trivial extension of α, i.e. $\hat{\alpha} = \alpha \circ (p \times 1_M)$, where $p : \mathbf{R}^{1|1} \to \mathbf{R}$ is the usual projection (which on functions is the inclusion of functions on $\mathbf{R}$ into forms on $\mathbf{R}$). Then

$$(\hat{\alpha}^* \nabla)_D(\hat{\alpha}^* s) = \theta(\alpha^* \nabla)_{\partial_t}(\alpha^* s) = \theta \alpha^*(\nabla_X s),$$

for all sections $s \in \Gamma(M, E)$. Since, as before, α^* is injective, the lift of D along α determines the lift of X given by the connection.

In this way, via the super parallel transport, we can lift *all* the vector fields on M to the derivations given by the connection, in other words, the super parallel transport *recovers the connection.*

4.5 Superparallel transport along superpaths

Let $(t,\theta) \in \mathbf{R}_+^{1|1}(S)$ be an S-point of $\mathbf{R}_+^{1|1}$. We define a super-analogue of the interval $I_t = [0,t]$ as follows:

Consider the triplet

$$S \xrightarrow[i_{(0,0)}]{} S \times \mathbf{R}^{1|1} \xleftarrow[i_{(t,\theta)}]{} S\ ,$$

with $i_{(0,0)}(s) = (s,0,0)$ and $i_{(t,\theta)}(s) = (s,t(s),\theta(s))$. Here $\mathbf{R}^{1|1}$ is endowed with the *standard metric structure* given by the odd vector field $D = \partial_\eta + \eta\partial_u$ in coordinates (u,η) on $\mathbf{R}^{1|1}$ (see Section 3.1). We denote this (family of) superinterval(s) by $I_{(t,\theta)}$.

Let x and y be S-points of M. A *superpath in M parametrized by the superinterval $I_{(t,\theta)}$ and with endpoints x and y* is an equivalence class of supercurves $c : S \times \mathbf{R}^{1|1} \to M$ with $c \circ i_{0,0} = c(0,0) = x$, respectively $c \circ i_{t,\theta} = c(t,\theta) = y$ such that $c \sim c'$ if there exists $\varepsilon > 0$ such that

$$c(u,\eta) = c'(u,\eta)$$

for all $(-\eta, 0) < (u,\eta) < (t+\eta,\theta)$. Here, " $<$ " is a partially defined order as follows: for $(t,\theta),\ (u,\eta) \in \mathbf{R}^{1|1}(S)$, we say

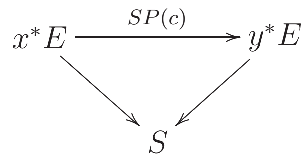
$$(u,\eta) < (t,\theta) \quad \text{if} \quad (t,\theta)(u,\eta)^{-1} \in \mathbf{R}_+^{1|1}(S).$$

Recall that $\mathbf{R}^{1|1}$ is a super Lie group- see Section 3.2- with the following group structure

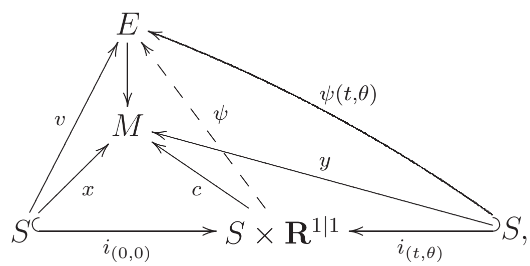
$$(t,\theta),\ (s,\eta) \longmapsto\ (t,\theta)(s,\eta) :=\ \ (t+s+\theta\eta, \theta+\eta).$$

In particular, for any supermanifold S, $\mathbf{R}^{1|1}(S) = \mathbf{SM}(S,\mathbf{R}^{1|1})$ is not just a set but a *group*. $\mathbf{R}_+^{1|1}$ is the open subsupermanifold in $\mathbf{R}^{1|1}$ whose reduced part is $\mathbf{R}_+ = (0,\ \infty)$. Such a superpath is denoted for short $c : I_{(t,\theta)} \to M$.

Let now $c : I_{(t,\theta)} \to M$ be a superpath in M. Then the connection ∇ on the bundle E will determine a vector bundle homomorphism



The map $SP(c)$ is given by a $\mathcal{C}^\infty(S)$-linear map $SP(c) : \Gamma(S, x^*E) \to \Gamma(S, y^*E)$ described by the following diagram



i.e. $SP(c)(v) = \psi(t,\theta)$, where ψ is the unique super parallel section of E along the supercurve c, such that $\psi(0,0) = v$. Since the solution ψ depends on the local data, it turns out that the map $SP(c)$ is well defined, i.e. it does not depend on a representative for the superpath $c : I_{(t,\theta)} \to M$. It is clearly a $\mathcal{C}^\infty(S)$-linear map, therefore it defines a bundle map $SP(c) : x^*E \to y^*E$.

The map SP satisfies the usual properties of a parallel transport map, i.e. it is compatible with gluing superpaths, and is invariant under reparametrizations (i.e. diffeomorphisms of superintervals that preserve the fiberwise conformal structure on $\mathbf{R}^{1|1}$).

4.6 Conclusion

In summary, we have: Let E be a super vector bundle over a supermanifold M endowed with a connection ∇. To any superpath $c : I_{t,\theta} \to M$ in M, we associate a bundle map $SP(c) : c^*_{0,0}E \to c^*_{t,\theta}E$, such that the following conditions hold:

1. The correspondence $c \mapsto SP(c)$ is smooth, and natural in S (see Lemma 4.2). Smoothness means the following: if c is a family of smooth superpaths parametrized by a supermanifold S, then the map $SP(c) : c_{0,0}^* E \to c_{t,\theta}^* E$ is a *smooth* bundle map over S.

2. (Compatibility under glueing) If $c : I_{t,\theta} \to M$ and $c' : I_{t',\theta'} \to M$ are two superpaths in M such that $c' \equiv c \circ R_{t,\theta}$ on some neighborhood $S \times U$ of $S \times (0,0) \hookrightarrow S \times \mathbf{R}^{1|1}$, with U an open subsupermanifold in $\mathbf{R}^{1|1}$ containing $(0,0)$, we have

$$SP(c' \cdot c) = SP(c') \circ SP(c),$$

where $c' \cdot c : I_{t'+t+\theta'\theta, \theta'+\theta} \to M$ is obtained from c and c' by glueing them along their "common endpoint", i.e.

$$(c' \cdot c)(s,u,\eta) = \begin{cases} c(s,u,\eta), & (u,\eta) < (t+\varepsilon, \theta) \\ \\ c'(s,(u,\eta)(t,\theta)^{-1}), & (t-\varepsilon,\theta) < (u,\eta). \end{cases}$$

(Here $R_{t,\theta} : S \times \mathbf{R}^{1|1} \to S \times \mathbf{R}^{1|1}$ is the right translation by (t,θ) in the $\mathbf{R}^{1|1}$-direction, i.e. $R_{t,\theta}(s,(u,\eta)) = (s,(u,\eta)(t,\theta))$.)

3. For any superpath $c : I_{t,\theta} \to M$, the bundle map $SP(c) : c_{0,0}^* E \to c_{t,\theta}^* E$ is an isomorphism, with inverse given by $PS(\overline{c}) : c_{t,\theta}^* E \to c_{0,0}^* E$, where $\overline{c} : I_{t,\theta} \to M$ is given by $\overline{c}(u,\eta) = c((u,\eta)^{-1}(t,\theta))$, and, for a superpath α in M, $PS(\alpha)$ denotes Q-parallel transport along α (see Remark 4.3).

4. (Invariance under reparametrization) Given $c : I_{t,\theta} \to M$ a superpath in M and $\varphi : I_{s,\eta} \to I_{t,\theta}$ a family of diffeomorphisms of superintervals that preserve the vertical distribution, we have

$$SP(c \circ \varphi) = SP(c).$$

Proof of (2). Since the construction of parallel transport is natural in S (see Lemma 4.2), it is enough to consider the case when S is "small" and c and c' map to a trivializing neighborhood $U \subseteq M$ for E, such that $E_{|U} \cong U \times \mathbf{R}^{p|q}$ and $\nabla = d + A$. If ψ is a super-parallel section along c with $\psi(0,0) = \psi_0$ and ψ' is super-parallel along c' with $\psi'(0,0) = \psi(t,\theta)$ then $\psi' \cdot \psi$ defined by

$$\psi' \cdot \psi(s,u,\eta) = \begin{cases} \psi(s,u,\eta), & (u,\eta) < (t+\varepsilon,\theta) \\ \\ \psi'(s,(u,\eta)(t,\theta)^{-1}), & (t-\varepsilon,\theta) < (u,\eta) \end{cases}$$

is a super-parallel section along $c' \cdot c$. (Observe that $\psi' \cdot \psi$ is well defined, by Prop. 4.1.) To show this, it is enough to prove the following

Lemma 4.1. *Let $c : S \times \mathbf{R}^{1|1} \to M$ be a superpath in M, and $A \in \Omega^1(M) \otimes End(\mathbf{R}^{p|q})^{ev}$. Let also $\psi : S \times \mathbf{R}^{1|1} \to \mathbf{R}^{p|q}$ be such that*

$$\partial_D \psi + (c^* A)(D)\psi = 0.$$

If $\bar{c} = c \circ R_{(t,\theta)}$ and $\bar{\psi} = \psi \circ R_{(t,\theta)}$, where $L_{(t,\theta)} : S \times \mathbf{R}^{1|1} \to S \times \mathbf{R}^{1|1}$ is left translation by $(t,\theta) \in \mathbf{R}^{1|1}(S)$ in the $\mathbf{R}^{1|1}$-direction, then

$$\partial_D \bar{\psi} + (\bar{c}^* A)(D)\bar{\psi} = 0.$$

Proof. Let φ be short for $R_{(t,\theta)}$. Then $\varphi^\sharp$ extends to $\mathbf{R}^{p|q}$-valued functions. Moreover, the vector field D is invariant under right translations, i.e. $\varphi_* D = D$, or, written differently, $D \circ \varphi^\sharp = \varphi^\sharp \circ D$. Applied to the $\mathbf{R}^{p|q}$-valued function ψ, this gives

$$\partial_D(\psi \circ \varphi) = (\partial_D \psi) \circ \varphi.$$

On the other hand,

$$\begin{aligned}(\bar{c}^*A)(D) &= (\varphi^*(c^*A))(D)\\ &= \varphi^\sharp(c^*A(\varphi_*D))\\ &= \varphi^\sharp(c^*A(D))\\ &= (c^*A(D))\circ\varphi.\end{aligned}$$

Therefore we have

$$\begin{aligned}\partial_D\bar{\psi} + (\bar{c}^*A)(D)\bar{\psi} &= \partial_D(\psi\circ\varphi) + \varphi^*(c^*A)(D)(\psi\circ\varphi)\\ &= \partial_D(\psi)\circ\varphi + \{(c^*A)(D)\psi\}\circ\varphi\\ &= 0.\end{aligned}$$

□

Proof of (3). Again, it is enough to assume that c maps to a trivializing neighborhood as before. Then ψ is super-parallel along c if

$$\partial_D\psi + (c^*A)(D)\psi = 0.$$

Consider the section $\bar{\psi}$ along $\bar{c}$ defined by $\bar{\psi}(s,u,\eta) = \psi(s,(u,\eta)^{-1}(t,\theta))$. Then $\bar{\psi}$ is Q-parallel along $\bar{c}$. To see this it is enough to prove the following

Lemma 4.2. *Let $c : S\times\mathbf{R}^{1|1} \to M$ be a superpath in M, and $A \in \Omega^1(M)\otimes End(\mathbf{R}^{p|q})^{ev}$. Let also $\psi : S\times\mathbf{R}^{1|1}\to\mathbf{R}^{p|q}$ be such that*

$$\partial_D\psi + (c^*A)(D)\psi = 0.$$

If $\bar{c} = c\circ R_{(t,\theta)}\circ I$ and $\bar{\psi} = \psi\circ R_{(t,\theta)}\circ I$, where $R_{(t,\theta)} : S\times\mathbf{R}^{1|1}\to S\times\mathbf{R}^{1|1}$ is right translation by $(t,\theta)\in\mathbf{R}^{1|1}(S)$ in the $\mathbf{R}^{1|1}$-direction, and $I : S\times\mathbf{R}^{1|1}\to S\times\mathbf{R}^{1|1}$:

$(s,u,\eta) \mapsto (s,-u,-\eta)$ *is the inversion map, then*

$$\partial_Q \bar{\psi} + (\bar{c}^* A)(Q)\bar{\psi} = 0.$$

Proof. Let us begin by showing that, via the inversion map $I : \mathbf{R}^{1|1} \to \mathbf{R}^{1|1} : (t,\theta) \mapsto (-t,-\theta)$, we have

$$I_* D = -Q.$$

For that, we need to show that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{C}^\infty(\mathbf{R}^{1|1}) & \xrightarrow{I^\sharp} & \mathcal{C}^\infty(\mathbf{R}^{1|1}) \\ {\scriptstyle -Q=-(\partial_\theta-\theta\partial_t)}\downarrow & & \downarrow{\scriptstyle D=\partial_\theta+\theta\partial_t} \\ \mathcal{C}^\infty(\mathbf{R}^{1|1}) & \xrightarrow{I^\sharp} & \mathcal{C}^\infty(\mathbf{R}^{1|1}). \end{array}$$

Following the diagram both ways we have

$$\begin{array}{ccc} t & \xmapsto{I^\sharp} & -t \\ {\scriptstyle -Q}\downarrow & & \downarrow{\scriptstyle D} \\ \theta & \xmapsto{I^\sharp} & -\theta \end{array} \qquad\qquad \begin{array}{ccc} \theta & \xmapsto{I^\sharp} & -\theta \\ {\scriptstyle -Q}\downarrow & & \downarrow{\scriptstyle D} \\ -1 & \xmapsto{I^\sharp} & -1. \end{array}$$

Coming back to the proof of the lemma, let us notice that $\bar{\psi}$ can be written

$$\bar{\psi} = I^\sharp R^\sharp \psi,$$

where R is short for $R_{(t,\theta)}$. Then

$$\begin{aligned} \partial_Q(\bar{\psi}) &= \partial_Q(I^\sharp R^\sharp \psi) \\ &= -I^\sharp \partial_D R^\sharp \psi \\ &= -I^\sharp R^\sharp \partial_D \psi, \end{aligned}$$

where the second equality is true by $\partial_Q I^\sharp = -I^\sharp \partial_D$ above and the third equality is true since D is a right-invariant vector field, i.e. $R^\sharp \partial_D = \partial_D R^\sharp$.

On the other side, we have

$$\begin{aligned}(\bar{c}^*A)(Q) &= (I^*R^*c^*A)(Q) \\ &= I^\sharp(R^*(c^*A)(I_*(Q))) \\ &= -I^\sharp(R^*(c^*A)(D)) \\ &= -I^\sharp R^\sharp((c^*A)(D)),\end{aligned}$$

where we used that the fact that

$$(f^*\omega)(Y) = f^\sharp(\omega(f_*Y)),$$

with $f : N \to M$ an arbitrary map of supermanifolds, $\omega \in \Omega^1(M)$, and $Y \in \mathfrak{X}(N)$. (The relation is true provided f_*Y exists, which is true in our cases.) Therefore

$$\begin{aligned}\partial_Q\bar{\psi} + (\bar{c}^*A)(Q)\bar{\psi} &= -I^\sharp R^\sharp \partial_D\psi - I^\sharp R^\sharp((c^*A)(D))(I^\sharp R^\sharp\psi) \\ &= -(\partial_D\psi + (c^*A)(D)\psi) \circ R \circ I \\ &= 0.\end{aligned}$$

The lemma is proved.

□

The conclusion of (3) follows. We can therefore state the main result of this chapter.

Theorem 4.3. *Any connection ∇ on a super vector bundle E over a supermanifold M gives rise to a correspondence*

$$I_{t,\theta} \xrightarrow{c} M \qquad \xmapsto{SP(\nabla)} \qquad c^*_{0,0}E \longrightarrow c^*_{t,\theta}E$$

satisfying the properties (1)-(4) above. Also, if $\nabla \neq \nabla'$ then $SP(\nabla) \neq SP(\nabla')$.

CHAPTER 5

VARIATIONS ON THE NOTION OF PARALLEL TRANSPORT

In this section we define a notion of A-parallel transport for the pair consisting of a connection and a bundle endomorphism A, and see that it converges (by an "inverse adiabatic limit" process) to the parallel transport of the connection. In particular, this means that the A-parallel transport is reparametrization invariant in the limit. The story is then carried over to the super case.

5.1 The parallel transport of (∇, A)

Let E be a vector bundle over the manifold M. Consider the pair consisting of a connection ∇ on E and $A \in \Gamma(M, End\ E)$ an endomorphism of E. We define a notion of parallel transport associated to (∇, A) as follows:

Let $c : I \to M$ be a path in M, defined on some compact interval I. A section $s \in \Gamma(I, c^*E)$ of E along c is called *A-parallel* if

$$(c^*\nabla)_{\partial_u} s + (c^*A)s = 0 \qquad \text{on} \quad I, \tag{5.1}$$

where u denotes the standard coordinate on I.

For $\lambda > 0$ let I_λ be an interval of length $\lambda|I|$, where $|I|$ is the length of I. Let $\varphi_\lambda : I_\lambda \to I$ be their rescaling diffeomorphism. For simplicity, consider $I = [0, T]$. Then $I_\lambda = [0, \lambda T]$ and $\varphi_\lambda : t \mapsto u = \frac{1}{\lambda}t$. Locally, $E \cong M \times \mathbf{R}^p$, if E is of rank p, and the connection ∇ can be written $\nabla = d + a$, for some $a \in \Omega^1(M) \otimes End(\mathbf{R}^p)$.

The first order differential equation (5.1) can then be written

$$\frac{ds}{du} + a(c'(u))s(u) + A(c(u))s(u) = 0.$$

The parallel transport along $\tilde{c} = c \circ \varphi_\lambda$ is given by

$$\frac{d\tilde{s}}{dt} + a(\tilde{c}'(t))\tilde{s}(t) + A(\tilde{c}(t))\tilde{s}(t) = 0.$$

If we write $\tilde{s} = s_\lambda \circ \varphi_\lambda$ then the last equation is equivalent to

$$\frac{ds_\lambda}{du} + a(c'(u))s_\lambda(u) + \lambda A(c(u))s_\lambda(u) = 0.$$

It is then clear that $\lim_{\lambda\to 0} s_\lambda = s_0$ is the parallel section along c in the parallel transport defined by the connection ∇. We write this symbolically as

$$P(\nabla, A) \longrightarrow P(\nabla).$$

5.2 The superparallel transport of (∇, A)

Let now E be a super vector bundle over a supermanifold M. Let (∇, A) be a pair consisting of a (grading preserving) connection ∇ on E and $A \in \Gamma(M, End\ E)$ an *odd* endomorphism of E. We define the notion of *super parallel transport corresponding to* (∇, A).

To do that, consider a family of supercurves $c : S \times \mathbf{R}^{1|1} \to M$ in M parametrized by the supermanifold S. A section $\psi \in \Gamma(c^*E)$ of E along c is *A-parallel* if it satisfies the equation

$$(c^*\nabla)_D\psi - (c^*A)\psi = 0. \tag{5.2}$$

This is again a "half-order" differential equation. In local coordinates, if $E_{|U} \cong U \times \mathbf{R}^{p|q}$, then $\nabla = d + a$, with $a \in \Omega^1(M, End E)^{odd}$ and the equation (5.2) can be written

$$\partial_D \psi + (c^* a)(D)\psi - (c^* A)\psi = 0,$$

where $D = \partial_\eta + \eta \partial_u$. Suppose for simplicity that (u, η) runs on the superinterval $I_{(T,\tau)}$, for $(T, \tau) \in \mathbf{R}_+^{1|1}(S)$ an S-superpoint of $\mathbf{R}_+^{1|1}$. Recall that $I_{(T,\tau)}$ is defined by the embeddings

$$S \xhookrightarrow[i_{(0,0)}]{} S \times \mathbf{R}^{1|1} \xhookleftarrow[i_{(T,\tau)}]{} S \, .$$

For $\lambda > 0$, let

$$\varphi_\lambda : I_{(\lambda T, \sqrt{\lambda}\tau)} \to I_{(T,\tau)} : \ (t, \theta) \mapsto (\frac{1}{\lambda} t, \frac{1}{\sqrt{\lambda}} \theta)$$

be the "rescaling" diffeomorphism that preserves the distribution $\mathcal{D}$. Then $\tilde{\psi}$ is A-parallel with respect to $\tilde{c} = c \circ \varphi_\lambda$ if

$$\partial_{\tilde{D}} \tilde{\psi} + (c^* a)(\tilde{D})\tilde{\psi} - (\tilde{c}^* A)\tilde{\psi} = 0,$$

where $\tilde{D} = \partial_\theta + \theta \partial_t$. If we write $\tilde{\psi} = \psi^\lambda \circ \varphi_\lambda = \varphi_\lambda^\sharp(\psi^\lambda)$ then the last equation can be rewritten

$$\partial_{\tilde{D}}(\varphi_\lambda^\sharp(\psi^\lambda)) + \varphi_\lambda^*(c^* a)(\tilde{D})\varphi_\lambda^\sharp(\psi^\lambda) - \varphi_\lambda^\sharp(c^* A)\varphi_\lambda^\sharp(\psi^\lambda) = 0,$$

Since $\varphi_{\lambda *}(\tilde{D}) = \frac{1}{\sqrt{\lambda}} D$ (see Lemma 4.1), which means that $\partial_{\tilde{D}} \varphi_\lambda^\sharp = \frac{1}{\sqrt{\lambda}} \varphi_\lambda^\sharp \partial_D$, the last equation is equivalent to

$$\frac{1}{\sqrt{\lambda}} \varphi_\lambda^\sharp \partial_D \psi^\lambda + \frac{1}{\sqrt{\lambda}} \varphi_\lambda^\sharp((c^* a)(D))\varphi_\lambda^\sharp(\psi^\lambda) - \varphi_\lambda^\sharp((c^* A)\psi^\lambda) = 0,$$

which gives

$$\partial_D \psi^\lambda + (c^* a)(D)\psi^\lambda - \sqrt{\lambda}(c^* A)\psi^\lambda = 0.$$

If we let $\lambda \to 0$ we see that $\psi^\lambda \longrightarrow \psi^0$, where ψ^0 is the superparallel section along c determined by the connection ∇. We conclude that the superparallel transport defined by (∇, A) converges in the "inverse adiabatic limit" to the superparallel transport of ∇, which is in particular invariant under reparametrization. Symbolically we write

$$SP(\nabla, A) \longrightarrow SP(\nabla).$$

CHAPTER 6

THE SUPER PARALLEL TRANSPORT OF SUPERCONNECTIONS

In this chapter we prove our main result: Any superconnection $\mathbb{A}$ on a $\mathbb{Z}/2$-graded vector bundle over a manifold gives rise to a super parallel transport $SP(\mathbb{A})$ which converges to the superparallel transport $SP(\mathbb{A}_1)$ of the connection part of the superconnection.

6.1 Preliminaries

Start with a $\mathbb{Z}/2$-graded vector bundle E over a *manifold* M, and consider a grading-preserving connection ∇ on E, together with an $End\ E$- valued form A on M, $A \in (\Omega^*(M, End\ E))^{odd}$. Combining these two pieces, we obtain a Quillen connection $\mathbb{A} = \nabla + A$ on E.

Recall the identification

$$\underline{\mathbf{SM}}(\mathbf{R}^{1|1}, M) = \underline{\mathbf{SM}}(\mathbf{R}, \Pi TM),$$

which for a supermanifold S gives

$$\mathbf{SM}(S \times \mathbf{R}^{1|1}, M) = \mathbf{SM}(S \times \mathbf{R}, \Pi TM).$$

Let $c:\ S\times\mathbf{R}^{1|1}\to M$ be a supercurve in M. Lift it to a supercurve $\tilde{c}$ in ΠTM as follows

$$\begin{array}{ccc} \Pi TM\times\mathbf{R}^{0|1} & \xrightarrow{T} & \Pi TM \\ {\scriptstyle \hat{c}\times 1}\uparrow & \overset{\tilde{c}}{\nearrow} & \downarrow{\scriptstyle \pi} \\ S\times\mathbf{R}^{1|1} & \xrightarrow[c]{} & M. \end{array}$$

The map π is given on functions by $\pi^\sharp:\mathcal{C}^\infty(M)\to\mathcal{C}^\infty(\Pi TM)=\ \Omega(M)$ is the inclusion of functions on M into the space of differential forms on M. The map T is the $\mathbf{R}^{0|1}$ action on ΠTM (see Lemma 3.3). The map $\hat{c}:S\times\mathbf{R}\to\Pi TM$ corresponds to $c:S\times\mathbf{R}^{1|1}\to M$ under the above identification. (**Note:** $\tilde{c}$ here stands for a *lift* of the curve c, and should not be confused for a reparametrization of c as in the previous chapter.)

Claim: The above diagram is commutative.

Proof of Claim: It is enough to show that the following diagram is commutative

$$\begin{array}{ccc} \Pi TM\times\mathbf{R}^{0|1} & \xrightarrow{T} & \Pi TM \\ {\scriptstyle \hat{\alpha}\times 1}\uparrow & & \downarrow{\scriptstyle \pi} \\ S\times\mathbf{R}^{0|1} & \xrightarrow[\alpha]{} & M, \end{array} \qquad (6.1)$$

for S an arbitrary supermanifold, and $\alpha:S\times\mathbf{R}^{0|1}\to M$ an arbitrary map. Here $\hat{\alpha}:S\to\Pi TM$ corresponds to α via Lemma 3.2. This translates into the following diagram being commutative

$$\begin{array}{ccc} \Omega^*(M)[\theta] & \xleftarrow{T^\sharp} & \Omega^*(M) \\ {\scriptstyle \hat{\alpha}^\sharp\otimes 1}\downarrow & & \uparrow{\scriptstyle \pi^\sharp} \\ \mathcal{C}^\infty(S)[\theta] & \xleftarrow[\alpha^\sharp]{} & \mathcal{C}^\infty(M). \end{array}$$

Recall (see the proof of Lemma 3.2) that if $\alpha^\sharp : f \mapsto a^\sharp(f) + \theta X_a(f)$, for $a \in M(S)$ and $X_a \in TM_a$, then $\hat{\alpha}^\sharp : \Omega^*(M) \to \mathcal{C}^\infty(S)[\theta]$ is determined by saying that $f \mapsto a^\sharp(f),\ df \mapsto X_a(f)$. Therefore, we have

$$f \stackrel{\pi^\sharp}{\longmapsto} f \stackrel{T^\sharp}{\longmapsto} f + \theta df \stackrel{\hat{\alpha}^\sharp \otimes 1}{\longmapsto} a^\sharp(f) + \theta X_a(f) = \alpha^\sharp(f).$$

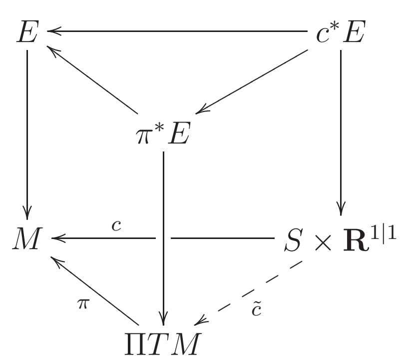
To complete the proof of the claim, it is enough to replace $S \mapsto S \times \mathbf{R}$ and $\alpha \mapsto c$ in the above considerations.

Remark 6.1. *It is not hard to check that the construction $c \mapsto \tilde{c}$ is natural in S, i.e.*

$$c \circ \widetilde{(\varphi \times 1)} = \tilde{c} \circ (\varphi \times 1),$$

for $\varphi : S' \to S$ an arbitrary map of supermanifolds.

Given the supercurve c, consider the pull-back diagram



where $\tilde{c}$ is as above. We call a section $\psi \in \Gamma(c^*E)$ of E along c *super-parallel* if it satisfies the equation

$$(c^*\nabla)_D \psi - (\tilde{c}^* A)\psi = 0.$$

This is again a "half-order" differential equation. It is equivalent to the equation

$$(\tilde{c}^*(\pi^*\nabla))_D \psi - (\tilde{c}^* A)\psi = 0.$$

Therefore $\psi \in \Gamma(c^*E)$ is super-parallel in the above sense if and only if ψ is A-parallel along the *lift* $\tilde{c}$ with respect to the pair $(\pi^*\nabla, A \in \Gamma(End(\pi^*E)))$ on the bundle $\pi^*E \to \Pi TM$, as defined in Section 5.2.

Analogously to Prop. 4.1, we have the following

Proposition 6.2. *Let $c : S \times \mathbf{R}^{1|1} \to M$ be a supercurve in the compact manifold M. Let $\psi_0 \in \Gamma(c^*_{0,0}E)$ be a section of E along $c_{0,0} : S \to M$. Then, there exists a unique super-parallel section (in the above sense) ψ of E along c, such that $\psi(0,0) = \psi_0$.*

An example. Let us see what the above construction gives in the case of $M = pt$. The bundle E together with the connection reduces in this case to a $\mathbb{Z}/2$-graded vector space V, and the bundle endomorphism valued form A reduces to an odd endomorphism $A \in End^1(V)$. We have the pull-back diagram

$$\begin{array}{ccc} E = E & \longleftarrow & E \times \mathbf{R}^{1|1} \\ \downarrow & & \downarrow \\ pt = pt & \xleftarrow[c=\tilde{c}]{} & \mathbf{R}^{1|1}. \end{array}$$

(It's enough to consider only this map, since the factor S doesn't play a role here.) The pull-back bundle is endowed with the trivial connection. The super parallel sections along c are therefore given by the equation

$$D\psi = A\psi.$$

Lemma 6.3. *The solutions of the above equation are given by*

$$(t, \theta) \mapsto e^{-tA^2 + \theta A} v,$$

for some v in V.

Proof. Indeed, we have:

$$\begin{aligned}(\partial_\theta + \theta\partial_t)e^{-tA^2+\theta A} &= (\partial_\theta + \theta\partial_t)[(1+\theta A)e^{-tA^2}] \\ &= Ae^{-tA^2} + \theta e^{-tA^2}(-A^2) \\ &= A(1+\theta A)e^{-tA^2} \\ &= Ae^{-tA^2+\theta A}\end{aligned}$$

where in the third equality we moved A past e^{-tA^2} without a sign change since e^{-tA^2} is even, and past θ with a change of sign, since both A and θ are odd. The lemma follows. □

The parallel transport therefore defines a map

$$\mathbf{R}^{1|1} \ni (t, \theta) \mapsto e^{-tA^2+\theta A} \in GL(V),$$

which is in fact a *supergroup homomorphism* $\mathbf{R}^{1|1} \to GL(V)$, since composition on $\mathbf{R}^{1|1}$, which preserves the vector field D, corresponds to composition (multiplication) on $GL(V)$. For a direct proof of this see [ST04].

6.2 Classical super-parallel sections

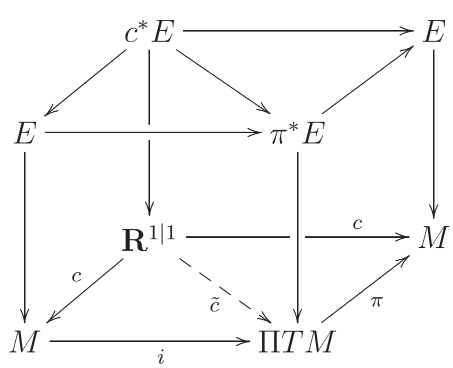
We want to address the question: What information do super-parallel sections carry along the "classical" superpaths? A *classical* superpath is a map $c : \mathbf{R}^{1|1} \to M$, which necessarily must factor through $\mathbf{R}$, since M is an ordinary manifold.

Let's consider first *constant* superpaths in M, i.e. superpaths $c : \mathbf{R}^{1|1} \to M$ that factor through pt, that is $c = c_x$ for some $x \in M$. We obtain a pull-back diagram

$$\begin{array}{ccc} E_x = E_x & \longleftarrow & E_x \times \mathbf{R}^{1|1} \\ \downarrow & & \downarrow \\ pt = pt & \xleftarrow[c=\tilde{c}]{} & \mathbf{R}^{1|1}, \end{array}$$

where E_x is the fiber over x. The bundle E pulls back to the trivial bundle and connection on E pulls back to the trivial connection. From the previous example, we see a supergroup homomorphism $\varphi : \mathbf{R}^{1|1} \to GL(E_x)$, whose infinitesimal generator (i.e. the image of the left-invariant vector field D via the differential of φ at $(0,0)$) is an odd endomorphism of E_x. This is exactly $A_{0x} = i_x^* A \in End(E_x)$, where $i_x : pt \to M$ is the map that takes a point to $x \in M$. Considering *all* the constant superpaths (parametrized by $x \in M$), we obtain a form $A_0 \in \Omega^0(M, EndE)$, which is exactly the 0-part of $A \in \Omega^*(M, EndE)$.

Consider now *arbitrary* superpaths $c : \mathbf{R}^{1|1} \to M$. Such a map will factor through $\mathbf{R}$. To describe the parallel transport along c, look at the diagram



The lift $\tilde{c}$ can also be written $\tilde{c} = i \circ c$, where i is the canonical inclusion map $M \to \Pi TM$, since the fermionic part of $\mathbf{R}^{1|1}$ maps trivially. The form $A \in \Omega^*(M, End\ E) \cong \Omega^*(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(EndE)$ determines the 0-form

$$A \in \Omega^0(\Pi TM, End(\pi^* E)) = \Gamma(\pi^*(End\ E)) \cong \mathcal{C}^\infty(\Pi TM) \otimes_{\mathcal{C}^\infty(M)} \Gamma(End\ E).$$

This pulls back to a 0-form via the map

$$\begin{array}{ccc} \Omega^0(\Pi TM, End(\pi^* E)) & \longrightarrow & \Omega^0(M, End(i^* \pi^* E)) \\ \| & & \| \\ \Omega^*(M, End\ E) & \longrightarrow & \Omega^0(M, End\ E). \end{array}$$

The last map is just projection on the 0-component, since

$$\Gamma(i^*\pi^* End\ E) = \Gamma(End\ E) \cong \mathcal{C}^\infty(M) \otimes_{\Omega^*(M)} \Gamma(\pi^* End\ E),$$

and the tensor product is via the projection on the 0-component $i^\sharp : \Omega^*(M) \to \mathcal{C}^\infty(M)$. Therefore the 0-form is exactly the 0-part $A_0 \in \Omega^0(M, End\ E)$ of A.

Lemma 6.1.

$$(c^*\nabla)_D = \partial_\theta + \theta(|c|^*\nabla)_{\partial_t},$$

where $|c| = c \circ i : \mathbf{R} \to M$ *is the "body" part of* c*, for* $i : \mathbf{R} \to \mathbf{R}^{1|1}$ *the standard inclusion of the reduced part.*

Proof. Let us first note that $c^*E = \ (|c|^*E) \times \mathbf{R}^{0|1}$, therefore $\Gamma(c^*E) = \Gamma(|c|^*E)[\theta]$. It is enough to consider the case of a bundle E over $\mathbf{R}$ with connection ∇, and would like to determine the lift of D under the pullback connection $\tilde{\nabla}$ via the map

$$p : \mathbf{R}^{1|1} \to \mathbf{R} : (t, \theta) \mapsto t.$$

We have $\tilde{\nabla}(s) = \nabla s$, and this extends to

$$\tilde{\nabla}(s_1 + \theta s_2) = \nabla s_1 + \theta \nabla s_2 + d\theta \otimes s_2,$$

therefore

$$\tilde{\nabla}_{\partial_\theta + \theta\partial_t}(s_1 + \theta s_2) = s_2 + \theta \nabla_{\partial_t} s_1,$$

hence the lemma. □

The equation of super-parallel transport

$$(c^*\nabla)_Q \psi - (\tilde{c}^*A)\psi = 0$$

becomes

$$(\partial_\theta + \theta(|c|^*\nabla)_{\partial_t})\psi - (c^*A_0)\psi = 0$$

or, if we write $\psi = s_1 + \theta s_2$ with $s_1, s_2 \in \Gamma(|c|^* E)$,

$$s_2 + \theta \nabla_t s_1 - (A_0 s_1 - \theta A_0 s_2) = 0,$$

since A_0 is odd. This in turn implies

$$s_2 = A_0 s_1 \qquad \text{and} \qquad \nabla_t s_1 + A_0^2 s_1 = 0.$$

Therefore we have

Lemma 6.2. *A section $\psi = s_1 + \theta s_2 \in \Gamma(c^* E)$ of E along $c : \mathbf{R}^{1|1} \to M$ is super-parallel with respect to (∇, A) if and only if its body part $|\psi| = s_1 \in \Gamma(|c|^* E)$ is parallel with respect to (∇, A_0^2) and uniquely determines ψ by the relation $s_2 = A_0 s_1$.*

6.3 A calculation

Let E be a $\mathbb{Z}/2$-graded vector bundle over M and ∇ a grading preserving connection on E. Consider the pull-back diagram

$$\begin{array}{ccc} E & \longleftarrow & \pi^* E \\ {\scriptstyle \nabla}\downarrow & & \downarrow{\scriptstyle \pi^*\nabla} \\ M & \underset{\pi}{\longleftarrow} & \Pi TM, \end{array}$$

where $\pi^* \nabla$ is the pull-back connection via the map π. By definition of the pull-back connection, the following diagram is commutative

$$\begin{array}{ccccc} \Gamma(E) & \xrightarrow{\nabla} & \Omega^1(M, E) & = & \Omega^1(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(E) \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \pi^*} & & \\ \Gamma(\pi^* E) & \xrightarrow[\pi^*\nabla]{} & \Omega^1(\Pi TM, \pi^* E) & = & \Omega^1(\Pi TM) \otimes_{\mathcal{C}^\infty(\Pi TM)} \Gamma(\pi^* E). \end{array}$$

Lemma 6.1.

$$(\pi^*\nabla)_\delta(\pi^* s) = \nabla s \in \Gamma(\pi^* E) \quad \textit{for} \quad s \in \Gamma(E),$$

where δ is the exterior differential d on forms $\Omega^*(M)$, viewed as a derivation on $\mathcal{C}^\infty(\Pi TM) = \Omega^*(M)$, i.e. a vector field on ΠTM.

Proof. We have by definition of the pull-back connection that

$$(\pi^*\nabla)(\pi^* s) = \ \pi^*(\nabla s).$$

If we write $\nabla s = \sum \omega_j \otimes s'_j \in \Omega^1(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(E)$, then

$$\pi^*(\nabla s) = \sum \omega_j \otimes \pi^* s'_j \in \Omega^1(\Pi TM) \otimes_{\mathcal{C}^\infty(\Pi TM)} \Gamma(\pi^* E).$$

Claim:

$$< \delta, \omega >= \varpi, \ \textit{for} \ \ \omega \in \pi^*(\Omega^1(M)) \hookrightarrow \Omega^1(\Pi TM),$$

where ϖ in the right hand side is the 1-form $\omega \in \Omega^1(M)$ understood as a function on ΠTM. Here $< \ , \ >$ stands for the pairing between vector fields and 1-forms (see Section 3.1).

Indeed, this is a local property, thus we can assume $M = \mathbf{R}^n$. Let us write $\omega = \sum f_i dx^i$, with $f_i \in \mathcal{C}^\infty(\mathbf{R}^n)$, and $\delta = \sum \theta^i \partial_i$, where θ^i is the form dx^i on $\mathbf{R}^n$ viewed as a function on $\Pi T\mathbf{R}^n$. Then

$$< \delta, \omega >=< \sum \theta^i \partial_i, \sum f_j dx^j >= \sum \theta^i f_i = \varpi.$$

Therefore,

$$< \delta, \pi^*(\nabla s) >=< \delta, \sum \omega_j \otimes \pi^* s'_j >= \sum \varpi_j \otimes s'_j = \nabla s \in \Gamma(\pi^* E),$$

hence the lemma.

□

Let $T : \Pi TM \times \mathbf{R}^{0|1} \to \Pi TM$ be the usual $\mathbf{R}^{0|1}$-action map. Then

$$T_*(\partial_\theta) = \delta \in \mathfrak{X}(\Pi TM). \tag{6.2}$$

Indeed, we need $\partial_\theta \circ T^\sharp = T^\sharp \circ \delta$. On one side, we have

$$\omega \xmapsto{T^\sharp} \omega + \theta\delta\omega \xmapsto{\partial_\theta} \delta\omega.$$

On the other side,

$$\omega \xmapsto{\delta} \delta\omega \xmapsto{T^\sharp} \delta\omega,$$

therefore (6.2) is true.

Consider now the "evaluation" map $ev : \Pi TM \times \mathbf{R}^{0|1} \to M$ defined as the composition $ev = \pi \circ T$, and let $\tilde{\nabla}$ denote the pull-back of the connection ∇ via the map ev. Let N be short for $\Pi TM \times \mathbf{R}^{0|1}$. Then, as a consequence of the above lemma, we have

Corollary 6.2.

$$\tilde{\nabla}_{\partial_\theta}(1 \otimes s) = 1 \otimes \nabla s \quad \textit{for} \quad s \in \Gamma(E),$$

where

$$1 \otimes s = ev^*(s) \in \Gamma(N, ev^*E) = \mathcal{C}^\infty(N) \otimes_{\mathcal{C}^\infty(M)} \Gamma(E)$$

and

$$1 \otimes \nabla s = T^*(\nabla s) \in \Gamma(N, T^*(\pi^*E)) = \mathcal{C}^\infty(N) \otimes_{\mathcal{C}^\infty(\Pi TM)} \Gamma(\pi^*E).$$

Proof. We have

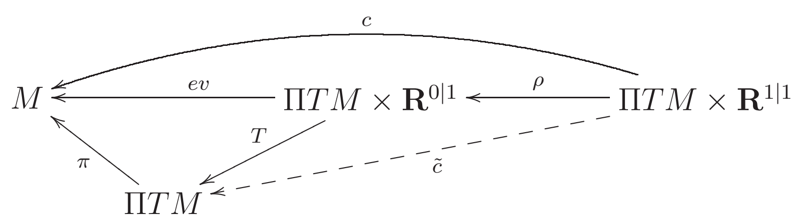
$$\begin{aligned}
\tilde{\nabla}_{\partial_\theta}(1 \otimes s) &= (T^*(\pi^*\nabla))_{\partial_\theta}(T^*(\pi^*s)) \\
&= T^*((\pi^*\nabla)_\delta(\pi^*s)) \\
&= T^*(\nabla s) \\
&= 1 \otimes s,
\end{aligned}$$

where the first and last equalities are true by definitions, the second is true by the definition of a pull-back connection and the fact that $T_*(\partial_\theta) = \delta$, as we've seen above. The third equality is true by the above lemma.

$\square$

6.4 Recovering the superconnection

In this section we recover the superconnections- i.e. pairs (∇, A) - from the parallel transport associated to them. We have already seen in Section 5.2 how the parallel transport of (∇, A) converges via an inverse adiabatic limit process to the parallel transport of ∇, which further recovers the connection ∇ - see Section 6.4. We are only left with recovering $A \in \Omega^*(M, End\ E)$. To do that, let us consider the following diagram



where ev is the "evaluation" map as in the previous section and $\rho = 1_{\Pi TM} \times p$, with $p : \mathbf{R}^{1|1} \to \mathbf{R}^{0|1}$ the natural projection map. Let us remark first that the lift of the curve $c = ev \circ \rho$ is the composition $T \circ \rho$. This holds by the naturality of lifts of supercurves (see Remark 6.1), and the fact that the lift of the "curve" ev is given by T (in diagram 6.1, if $\alpha = ev$ then $\widehat{ev} = 1_{\Pi TM}$).

By definition, a section $\psi \in \Gamma(c^*E)$ of E along c is parallel if

$$(c^*\nabla)_D\psi - (\rho^*T^*A)\psi = 0.$$

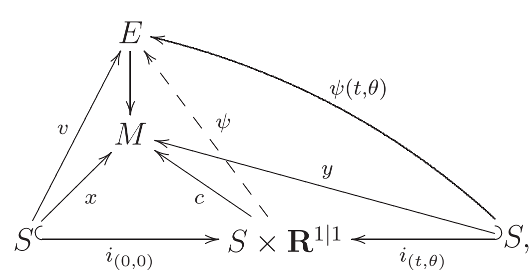
We therefore know the operator

$$(c^*\nabla)_D - \rho^*T^*A : \ \Gamma(c^*E) \longrightarrow \Gamma(c^*E)$$

on parallel sections. But that is enough to determine it, since the parallel sections generate $\Gamma(c^*E)$ as a $\mathcal{C}^\infty(\Pi TM \times \mathbf{R}^{1|1})$- module. On the other hand, we know the operator $(c^*\nabla)_D : \Gamma(c^*E) \longrightarrow \Gamma(c^*E)$, since we know the connection ∇. In this manner we determine the linear map $\rho^* T^* A$. Since both ρ^* and T^* are injective, this uniquely determines A. In this manner, we recovered the superconnection (∇, A) from the associated parallel transport.

6.5 Super-parallel transport along superpaths

Let $c : I_{t,\theta} \to M$ be a superpath in M with $c(0,0) = x$ and $c(t,\theta) = y$. Then the super-parallel transport of ∇ and $A \in \Omega^*(M, EndE)$ will determine a bundle homomorphism $SP(c) : x^*E \to y^*E$. This is defined as in Section 4.5 by a $\mathcal{C}^\infty(S)$-linear map $SP(c) : \Gamma(S, x^*E) \to \Gamma(S, y^*E)$ described by the diagram



i.e. $SP(c)(v) = \psi(t,\theta)$, where ψ is the unique super parallel section with respect to the pair (∇, A) of E along the supercurve c, such that $\psi \circ i_{(0,0)} = v$.

6.6 Main Theorem

We are now in the position to state our main theorem.

Theorem 6.1. *Let E be a $\mathbb{Z}/2$-graded vector bundle over a manifold M. Let ∇ be a grading preserving connection on E and $A \in \Omega^*(M, EndE)^{odd}$. The pair (∇, A)*

gives rise to a correspondence $SP = SP(\nabla, A)$

$$I_{t,\theta} \xrightarrow{c} M \qquad \xmapsto{SP} \qquad c^*_{0,0}E \longrightarrow c^*_{t,\theta}E$$

such that:

1. *The correspondence* $c \mapsto SP(c)$ *is smooth, and natural in* S *(see Lemma 4.2).*

2. *(Compatibility under glueing) If* $c : I_{t,\theta} \to M$ *and* $c' : I_{t',\theta'} \to M$ *are two superpaths in* M *such that* $c' \equiv c \circ R_{t,\theta}$ *on some neighborhood* $S \times U$ *of* $S \times (0,0) \hookrightarrow S \times \mathbf{R}^{1|1}$*, with* U *an open subsupermanifold in* $\mathbf{R}^{1|1}$ *containing* $(0,0)$*, we have*

$$SP(c' \cdot c) = SP(c') \circ SP(c),$$

   where $c' \cdot c : I_{t'+t+\theta'\theta, \theta'+\theta} \to M$ *is obtained from* c *and* c' *by glueing them along their "common endpoint".*

Moreover, if $\nabla \neq \nabla'$ *or* $A \neq A'$ *then* $SP(\nabla, A) \neq SP(\nabla', A')$*. Also,* $SP(\nabla, A)$ *converges in the inverse adiabatic limit to* $SP(\nabla)$*.*

Proof. The properties (1) and (2) are clear from the construction of the super-parallel transport of the pair (∇, A). Two different such pairs (superconnections) give rise to two different super-parallel transports, since the parallel transport recovers the superconnection, according to Section 6.4. The inverse adiabatic limit process is described in Section 5.2. □

Remark 6.2. *In particular, a superconnection* $\mathbb{A}$ *on the bundle* E *over* M *(in the sense of Quillen) gives rise to a super-parallel transport based on* M*, namely consider the super-parallel transport associated to the pair* $(\nabla = \mathbb{A}_1, A = \sum_{i \neq 1} \mathbb{A}_i)$*.*

BIBLIOGRAPHY

[AHS01] M. Ando, M. J. Hopkins, and N. P. Strickland. Elliptic spectra, the Witten genus and the theorem of the cube. *Invent. Math.*, 146(3):595–687, 2001.

[AHS04] Matthew Ando, Michael J. Hopkins, and Neil P. Strickland. The sigma orientation is an H_∞ map. *Amer. J. Math.*, 126(2):247–334, 2004.

[Ati88] Michael Atiyah. Topological quantum field theories. *Inst. Hautes Études Sci. Publ. Math.*, (68):175–186 (1989), 1988.

[BBHR91] Claudio Bartocci, Ugo Bruzzo, and Daniel Hernández Ruipérez. *The geometry of supermanifolds*, volume 71 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1991.

[BDR04] Nils A. Baas, Bjorn Ian Dundas, and John Rognes. Two-vector bundles and forms of elliptic cohomology. In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 18–45. Cambridge Univ. Press, Cambridge, 2004.

[BGV92] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators*, volume 298 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[DM99] Pierre Deligne and John W. Morgan. Notes on supersymmetry (following Joseph Bernstein). In *Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997)*, pages 41–97. Amer. Math. Soc., Providence, RI, 1999.

[Hel01] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[HK04] P. Hu and I. Kriz. Conformal field theory and elliptic cohomology. *Adv. Math.*, 189(2):325–412, 2004.

[Hop02] M. J. Hopkins. Algebraic topology and modular forms. In *Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002)*, pages 291–317, Beijing, 2002. Higher Ed. Press.

[KN63] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol I.* Interscience Publishers, a division of John Wiley & Sons, New York-Lond on, 1963.

[Kos77] Bertram Kostant. Graded manifolds, graded Lie theory, and prequantization. In *Differential geometrical methods in mathematical physics (Proc. Sympos., Univ. Bonn, Bonn, 1975)*, pages 177–306. Lecture Notes in Math., Vol. 570. Springer, Berlin, 1977.

[Lei80] D. A. Leites. Introduction to the theory of supermanifolds. *Uspekhi Mat. Nauk*, 35(1(211)):3–57, 255, 1980.

[Man88] Yuri I. Manin. *Gauge field theory and complex geometry*, volume 289 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1988.

[Mar05] Elke Markert. *Connective 1-dimensional field theories*. Ph.D. Thesis. Univ. of Notre Dame, 2005.

[MP02] Marco Mackaay and Roger Picken. Holonomy and parallel transport for abelian gerbes. *Adv. Math.*, 170(2):287–339, 2002.

[Och87] Serge Ochanine. Sur les genres multiplicatifs définis par des intégrales elliptiques. *Topology*, 26(2):143–151, 1987.

[Poo81] Walter A. Poor. *Differential geometric structures*. McGraw-Hill Book Co., New York, 1981.

[Rez01] Charles Rezk. *Suplementary Notes for Math 512*. Notes based on a course given at Northwestern U. on topological modular forms. 2001.

[Šan80] V. N. Šander. Vector fields and differential equations on supermanifolds. *Funktsional. Anal. i Prilozhen.*, 14(2):91–92, 1980.

[Seg88] Graeme Segal. Elliptic cohomology (after Landweber-Stong, Ochanine, Witten, and others). *Astérisque*, (161-162):Exp. No. 695, 4, 187–201 (1989), 1988.

[Seg04] Graeme Segal. The definition of conformal field theory. In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 421–577. Cambridge Univ. Press, Cambridge, 2004.

[ST04] Stephan Stolz and Peter Teichner. What is an elliptic object? In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 247–343. Cambridge Univ. Press, Cambridge, 2004.

[Var04] V. S. Varadarajan. *Supersymmetry for mathematicians: an introduction*, volume 11 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2004.

[Wit86] Edward Witten. The index of the Dirac operator on loop space. In *Elliptic curves and modular forms in alg.top.*, volume 1326 of *LNM*, pages 161–181. Springer, 1986.