THE NOTION OF VERTEX OPERATOR COALGEBRA: A CONSTRUCTION AND GEOMETRIC INTERPRETATION

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Abstract

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The notion of vertex operator coalgebra is presented, which corresponds to the family of correlation functions modeling one string propagating in space-time splitting into n strings in conformal field theory. Specifically, we describe the category of geometric vertex operator coalgebras, whose objects have comultiplicative structures meromorphically induced by conformal equivalence classes of the worldsheets swept out by propagating strings. We then show that this category is isomorphic to the category of vertex operator coalgebras, which is defined in the language of formal algebra with a generalized Jacobi-like identity. This notion is in some sense dual to the notion of vertex operator algebra. We also prove that any vertex operator algebra equipped with a non-degenerate, Virasoro preserving, bilinear form gives rise to a corresponding vertex operator coalgebra. Finally, we explicitly calculate the vertex operator coalgebra structure and unique bilinear form for the Heisenberg algebra case, which corresponds to considering free bosons in conformal field theory.

CONTENTS

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CHAPTER 1

INTRODUCTION

In this thesis, we define the notion of a vertex operator coalgebra as motivated through the geometry of conformal field theory. Specifically, after defining the category whose objects are vertex operator coalgebras (VOCs), we show that this category is isomorphic to the category whose objects are geometrically defined structures called geometric vertex operator coalgebras (GVOCs). In addition, we will provide a large family of examples of VOCs including VOCs generated by Heisenberg algebras. The definition of a geometric vertex operator coalgebra is motivated in a similar way to that of a geometric vertex operator algebra ([H1], [H2]) in that both arise from the geometry underlying conformal field theory, specifically closed string theory, in physics. Conceptually, string theory seeks to unify all the fundamental forces in nature into a single theory by modeling particles as one-dimensional objects (as opposed to point-particles) moving through space-time. These objects, called strings, sweep out two-dimensional manifolds, called worldsheets, in space-time that model the interactions of particles.

Closed string theory focuses specifically on strings that begin and end at the same point, that is, on loops. Their study, along with the study of the algebraic structure they induce began in physics (cf. [BPZ], [FS], [S] and [V]). Physicists introduced the notion of vertex operators in order to write down the expectation values of certain particle interactions. These vertex operators were independently discovered in mathematics in the study of representations of affine Lie algebras [LW]. Vertex algebras were first introduced in [Bo] and describe constraints on the multiplicative structure induced by modeling two strings combining into one as they travel through space-time. The notion of a vertex operator algebra (VOA) was then introduced in [FLM], which interpreted vertex algebras in a formal calculus setting with a generating function greatly generalizing the classical Jacobi identity and, perhaps most significantly, added a compatible conformal structure.

Although vertex operator algebras had origins in the geometry of string theory, a rigorous correspondence between the algebraic formalism and the geometric motivation was not completed until Huang introduced the notion of a geometric vertex operator algebra (GVOA) in [H1] and [H2]. Motivated by the physicists' study of the geometric interaction (specifically chiral interaction) of $n \in \mathbb{N}$ closed strings combining to form one closed string, Huang defined geometric vertex operator algebras in the context of a mathematical description of genus-zero Riemann surfaces with tubes up to conformal equivalence, along with a sewing operation on these objects. He also proved that the category of GVOAs was isomorphic to the category of VOAs, finalizing the rigorous correspondence between VOAs and the geometry of genus-zero worldsheets. The interactions corresponding to the worldsheets that Huang focused on are important special cases of interactions that may occur in a conformal field theory.

Another way that Huang's definition of GVOAs proved fruitful, was that it showed a VOA could be described in the language of operads, a language first developed by May in [M] to discuss iterated loop spaces. Specifically, Huang's GVOA corresponds to describing the partial operad of equivalence classes of compact genus-zero Riemann surfaces with n incoming punctures, one outgoing puncture and local coordinates vanishing at each puncture, then taking the meromorphic associative algebra associated to $\mathbb{C}$-extensions of that partial operad [HL1]. Using this interpretation, attaching or "sewing" two worldsheets together corresponds to composing operators in an operad defined in the category of complex vector spaces.

Prior to VOAs being solidly tied to their foundations in the geometry of conformal field theory, significant applications for VOAs had been emerging outside of conformal field theory. VOAs are now known to have deep connections to modular functions, have contributed to the representation theory of the Monster finite simple group and are fundamental to the study of representations of infinite-dimensional Lie algebras. (See the introduction of [FLM] for a more thorough exposition on connections and applications.)

Modules of VOAs quickly became central to the study of conformal field theory since sums and tensor products of these modules are required to provide a full modular invariant, monodromy free conformal field theory. In order to understand actual constructions in this context and provide a global setting, vertex tensor categories where introduced and developed in [HL2-HL5,H3]. These categories have also been conjectured to hold answers to other longstanding paradoxes related to group theory and monstrous moonshine [HL5]. Understanding appropriate coalgebraic structure is the first step toward understanding bialgebraic and Hopf algebraic notions in vertex tensor categories, which should lead to a new interpretation of the tensor product of VOA modules, just as Hopf algebras provide a module tensor theory for Lie algebras.

Geometric vertex operator coalgebras (GVOCs) are designed to model genus-zero worldsheets corresponding to one incoming closed string splitting into n outgoing closed strings. Similar to the GVOA case, this amounts to describing the partial operad of equivalence classes of compact genus-zero Riemann surfaces with one incoming and n outgoing punctures, then investigating the algebraic structure it induces on complex vector spaces, i.e. determining the meaning of a coalgebra over this partial operad. The GVOC structure is the natural coalgebraic structure with respect to GVOAs and should play a role in the development of the theory of vertex tensor categories as mentioned above.

The main thrust of this thesis, however, is to understand VOCs from an algebraic and geometric standpoint. To that end, the notion of a vertex operator coalgebra is defined (purely in terms of vector spaces, formal variables and formal operators) and is proven to be categorically isomorphic to the notion of GVOC. Generating a formal algebraic description in the VOA case has facilitated a more concrete understanding of the structure and allowed for the construction of numerous examples. Similarly the algebraic description of VOCs allows us to describe a family of examples and opens the door to a deeper understanding of a conformal field theory's algebraic structure. One might reasonably wonder whether the definition of VOC might be arrived at through purely algebraic means. But "standard" dualizing would not produce bounding of formal variables in opposite directions that arise in VOCs - specifically, truncation produces only finitely many positive powers of the given formal variable in a VOC, while the counit property produces only non-negative powers (in VOAs everything is truncated from below). Additionally, results of weak commutativity coming from the VOC Jacobi identity reveal an important distinction from that of the VOA Jacobi identity [Hub].

In addition, since VOAs are of interest outside of string theory, it is natural to hope that VOCs, once well understood, might introduce new connections to broader mathematics as well.

CHAPTER 2

THE GEOMETRY OF SPHERES WITH TUBES

We begin by investigating the geometry of spheres with tubes. Since our main motivation in this thesis is studying closed strings propagating through space-time it is essential that we precisely define the types of structures that model this behavior. This description will closely follow [H2] since that work contains the description of the partial operad underlying GVOAs (motivated by string theory) and also because we would like GVOCs to be compatible with that structure. In Section 2.1 we begin by defining what we mean by a sphere with tubes. In Section 2.2 we describe the process for attaching, or “sewing”, two of these spheres.

2.1 Defining spheres with tubes

Geometric vertex operator algebras or coalgebras are defined using the moduli space (under conformal equivalence) of compact genus-zero Riemann surfaces with punctures and local coordinates vanishing at each puncture. Vafa first observed that having a puncture on a Riemann surface together with local coordinates that vanish at that puncture is conformally equivalent to having a half-infinite tube attached to the Riemann surface (cf. [V], [H2]). We will begin by defining exactly what we mean by a genus-zero Riemann surface with punctures and local coordinates vanishing at the punctures, or equivalently, a genus-zero worldsheet.

Consider a compact genus-zero Riemann surface, by which we mean a compact connected genus-zero one-dimensional complex manifold. In [H2] and in this paper these surfaces are simply referred to as "spheres". The reason for the use of the word spheres is that any compact Riemann surface of genus-zero is complex analytically isomorphic to the standard Riemann sphere, i.e. $\mathbb{C} \cup \{\infty\}$ with the standard complex structure (cf. [A], [FK]). Since we will eventually be concerned only with conformal equivalence classes of Riemann surfaces, in Chapter 3 we will pick canonical representatives of the equivalences classes under conformal equivalence and these canonical representatives will be Riemann spheres.

By the term *oriented puncture* we mean the selection of a point of a sphere together with an element of {+, -}. Given an oriented puncture p, a *local analytic coordinate chart vanishing at p* is a pair (U, Φ) where U is an open neighborhood of p in the sphere, called the *local coordinate neighborhood*, and $\Phi : U \to \mathbb{C}$ is an injective analytic map, such that $\Phi(p) = 0$, called the *local coordinate map*. The term *tube centered at p* is used interchangeably with the term local analytic coordinate chart vanishing at p and may by thought of as the path of a single closed string propagating through space-time. Tubes centered at negatively oriented punctures represent half-infinite tubes swept out by outgoing strings, while tubes centered at positively oriented punctures represent half-infinite tubes swept out by incoming strings.

Spheres with m tubes centered at negatively oriented punctures and n tubes centered at positively oriented punctures are said to be of *type* (m, n). All oriented punctures are required to be distinct and have an ordering with negatively oriented punctures coming first. In [H2], the focus is primarily on spheres of type $(1, m)$, with $m \in \mathbb{N}$. In this work, we will discuss spheres of type $(1, m)$, $m \in \mathbb{N}$, called *spheres with incoming tubes*, and of type $(m, 1)$, $m \in \mathbb{N}$, called *spheres with outgoing* *tubes*, but will eventually focus on the latter.

For the moment, however, we will discuss all genus-zero worldsheets. Using the above framework, we can denote a sphere (or compact Riemann surface of genus-zero) with m outgoing punctures, n incoming punctures and corresponding vanishing local coordinates by

$$(S; p_{-m}, \ldots, p_{-1}, p_1, \ldots, p_n; (U_{-m}, \Phi_{-m}), \ldots, (U_{-1}, \Phi_{-1}), (U_1, \Phi_1), \ldots, (U_n, \Phi_n))$$

where $m, n \in \mathbb{N}$, S is a compact genus-zero Riemann surface, p_i is a point in S with the sign of the index corresponding to the orientation of the puncture, and (U_i, Φ_i) is a local analytic coordinate chart vanishing at p_i for $i = -m, \ldots -1, 1, \ldots, n$. The terms *sphere with tubes*, *genus-zero worldsheet*, and compact Riemann surface of genus-zero with punctures and local coordinates vanishing at the punctures are used interchangeably in the literature to refer to this structure. (The reader should be aware that while this notation is technically necessary to establish a moduli space of genus-zero worldsheets, it will not be used once equivalence classes under conformal equivalence are defined and canonical representatives for each equivalence class are established.)

Let

$$\Sigma_1 = (S_1; p_{-m}, \ldots, p_{-1}, p_1, \ldots, p_n;$$
$$(U_{-m}, \Phi_{-m}), \ldots, (U_{-1}, \Phi_{-1}), (U_1, \Phi_1), \ldots, (U_n, \Phi_n))$$

be a sphere of type (m, n) and

$$\Sigma_2 = (S_2; q_{-k}, \ldots, q_{-1}, q_1, \ldots, q_\ell;$$
$$(V_{-k}, \Psi_{-k}), \ldots, (V_{-1}, \Psi_{-1}), (V_1, \Psi_1), \ldots, (V_\ell, \Psi_\ell))$$

be a sphere of type (k, ℓ). We say that Σ_1 and Σ_2 are *conformally equivalent* if

$m = k$, $n = \ell$ and there is a complex analytic isomorphism $F : S_1 \to S_2$ such that $F(p_i) = q_i$ and $\Phi_i = \Psi_i \circ F$ in some neighborhood of p_i for $i = -m, \ldots, -1, 1, \ldots, n$.

2.2 Sewing spheres with tubes

A fundamental property of the interactions of strings in conformal field theory is that certain interactions should compose naturally. In geometry we model composition by combining two Riemann surfaces into a single Riemann surface. This procedure is rigorously established in [H2] for "sewing", or attaching, two spheres of type $(1, m)$ and $(1, n)$, respectively. However, Huang's description never uses the orientation on the punctures that remain unsewn, allowing the argument to be directly generalized to sewing one incoming tube of a sphere of type (m, n) to one outgoing tube of a sphere of type (k, ℓ) for $m, \ell \in \mathbb{N},\ n, k \in \mathbb{Z}_+$. Following [H2], we will describe the conditions under which such a sewing may occur and describe the resulting sphere with tubes.

We will use $B^r \subset \mathbb{C}$ and $\overline{B}^r \subset \mathbb{C}$ to denote the open and closed discs of radius r centered at the origin.

Let

$$\Sigma_1 = (S_1; p_{-m}, \ldots, p_{-1}, p_1, \ldots, p_n; \\ (U_{-m}, \Phi_{-m}), \ldots, (U_{-1}, \Phi_{-1}), (U_1, \Phi_1), \ldots, (U_n, \Phi_n))$$

be a sphere of type (m, n), for $m \in \mathbb{N}$, $n \in \mathbb{Z}_+$ and

$$\Sigma_2 = (S_2; q_{-k}, \ldots, q_{-1}, q_1, \ldots, q_\ell; \\ (V_{-k}, \Psi_{-k}), \ldots, (V_{-1}, \Psi_{-1}), (V_1, \Psi_1), \ldots, (V_\ell, \Psi_\ell))$$

be a sphere of type (k, ℓ), for $k \in \mathbb{Z}_+$, $\ell \in \mathbb{N}$. Choose integers $1 \le i \le n$ and $1 \le j \le k$. We say that the *i-th tube of Σ_1 can be sewn with the $-j$-th tube of Σ_2* if there exists $r \in \mathbb{R}_+$ such that

$$\overline{B}^r \subset \Phi_i(U_i), \tag{2.2.1}$$

$$\overline{B}^{1/r} \subset \Psi_{-j}(V_{-j}), \tag{2.2.2}$$

p_i is the only puncture in $\Phi_i^{-1}(\overline{B}^r)$, and q_{-j} is the only puncture in $\Psi_{-j}^{-1}(\overline{B}^{1/r})$. We also say that the i-th tube of Σ_1 can be sewn with the $-j$-th tube of Σ_2 in the case that Equations (2.2.1) and (2.2.2) do not hold for the domains of Φ_i and Ψ_{-j}, but there exists $r \in \mathbb{R}_+$ such that the domain of Φ_i^{-1} may be analytically extended to $\overline{B}^r$ without $\Phi_i^{-1}(\overline{B}^r)$ containing punctures other than p_i, and the domain of Ψ_{-j}^{-1} may be analytically extended to $\overline{B}^{1/r}$ without $\Psi_{-j}^{-1}(\overline{B}^{1/r})$ containing punctures other that q_{-j}. From such a Σ_1 and Σ_2 we obtain a sphere with tubes of type $(m+k-1, n+\ell-1)$ as follows. First, choose r_1, r_2 real numbers such that $0 < r_2 < r < r_1$, $\overline{B}^{r_1} \subset \mathrm{im}(\Phi_i)$, $\overline{B}^{1/r_2} \subset \mathrm{im}(\Psi_{-j})$, $\Phi_i^{-1}(\overline{B}^{r_1})$ contains no punctures of Σ_1 besides p_i, and $\Psi_{-j}^{-1}(\overline{B}^{1/r_2})$ contains no punctures of Σ_2 besides q_{-j}. Then remove $\Phi_i^{-1}(\overline{B}^{r_2})$ from S_1, remove $\Psi_{-j}^{-1}(\overline{B}^{1/r_1})$ from S_2, and identify the collars surrounding these two removed neighborhoods via the map $\Psi_{-j}^{-1}(\frac{1}{\Phi_i(w)})$. In regard to ordering the punctures, the outgoing punctures are ordered by inserting all ordered outgoing punctures of S_1 between the ordered outgoing punctures q_{-j-1} and q_{-j+1} of S_2, while the incoming punctures are ordered by inserting all incoming punctures of S_2 between the incoming punctures p_{i-1} and p_{i+1} of S_1. Explicitly, after sewing we have the sphere

$$S_3 = ((S_1 \smallsetminus \Phi_i^{-1}(\overline{B}^{r_2}) \sqcup (S_2 \smallsetminus \Psi_{-j}^{-1}(\overline{B}^{1/r_1}))/\sim$$

where (using the notation of [V]) $\sqcup$ is the disjoint union and $\sim$ is the equivalence relation, preserving complex structure, given by $p \sim q$ if and only if $p = q$ or $p \in \Phi_i^{-1}(\overline{B}^{r_1}) \smallsetminus \Phi_i^{-1}(\overline{B}^{r_2})$, $q \in \Psi_{-j}^{-1}(\overline{B}^{1/r_2}) \smallsetminus \Psi_{-j}^{-1}(\overline{B}^{1/r_1})$ and $\Psi_{-j}^{-1}(\frac{1}{\Phi_i(p)}) = q$. Thus we define the sewing of the i-th tube of Σ_1 with the $-j$-th tube of Σ_2 to be the sphere S_3 with the punctures

$$q_{-k}, \ldots, q_{-j-1}, p_{-m}, \ldots, p_{-1}, q_{-j+1}, \ldots, q_{-1}, p_1, \ldots, p_{i-1}, q_1, \ldots, q_\ell, p_{i+1}, \ldots, p_n$$

and the local coordinate maps restricted appropriately; that is, for each puncture p of Σ_1 remove $\Phi_i^{-1}(\overline{B}^{r_1})$ from the local coordinate neighborhood and restrict the local coordinate map at p to this new local coordinate neighborhood, and do similarly for each puncture q in Σ_2 and its local coordinate chart. This sewing is independent of the choice of r, r_1, r_2 ([H2]).

CHAPTER 3

CONFORMAL EQUIVALENCE ON SPHERES WITH TUBES

At first glance, spheres with tubes might appear to be exactly the right picture for modeling closed strings in space-time. However, by definition conformal field theories assume invariance of interactions under conformal transformations. Thus two spheres with tubes which possess an invertible conformal map between them will result in equivalent correlation functions for the resulting particle interaction.

As a result, we study conformal equivalence classes of spheres with tubes. Recall from Section 2.1 that two spheres S_1 and S_2 are conformally equivalent if they both have the same number of incoming and outgoing punctures and if there is a complex analytic isomorphism $F : S_1 \to S_2$ mapping punctures to punctures which preserves the order of punctures and maps local coordinate charts to local coordinate charts in some neighborhood of each puncture. This gives us a natural sense of conformal equivalence classes of spheres with tubes that respects the type of each sphere. We will first review the implications of this equivalence class structure on spheres of type $(1, n)$, for $n \in \mathbb{N}$, called *spheres with incoming tubes*, as covered in [H2] and then explore the analogous implications for *spheres with outgoing tubes*, i.e. spheres of type $(n, 1)$, for $n \in \mathbb{N}$.

3.1 The moduli space of spheres with incoming tubes

First, we follow Huang's work ([H1], [H2]) describing canonical representatives for each conformal equivalence class of spheres of type $(1, n)$, for $n \in \mathbb{Z}_+$. (Spheres of type $(1, 0)$ will be dealt with separately.)

For any $r \in \mathbb{R}_+$ and $z \in \mathbb{C}^\times$, let

$$B_0^r = \{w \in \widehat{\mathbb{C}} |\ |w| < r\},$$

$$B_z^r = \{w \in \widehat{\mathbb{C}} |\ |z - w| < r\},$$

$$B_\infty^r = \{w \in \widehat{\mathbb{C}} |\ |1/w| < r\}.$$

Recall that the reason we refer to any compact genus-zero Riemann surface as a sphere is that each one is complex analytically isomorphic to the standard sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Furthermore, we have the following proposition.

Proposition 3.1.1. *Any sphere with tubes of type $(1, n)$, for $n \in \mathbb{Z}_+$, is conformally equivalent to a sphere of the form*

$$(\widehat{\mathbb{C}}; z_{-1}, z_1, \dots, z_n; (B_{z_{-1}}^{r_{-1}}, \Phi_{-1}), (B_{z_1}^{r_1}, \Phi_1), (B_{z_2}^{r_2}, \Phi_2), \dots, (B_{z_n}^{r_n}, \Phi_n)) \tag{3.1.1}$$

where $z_{-1} = \infty$, $z_n = 0$, $z_i \in \mathbb{C}^\times$ for $i = 1, \dots, n-1$, satisfying $z_i \neq z_j$ for $i \neq j$, $r_i \in \mathbb{R}_+$ for $i = -1, 1, 2, \dots, n$ and $\Phi_{-1}, \Phi_1, \Phi_2, \dots, \Phi_n$ are analytic on $B_\infty^{r_{-1}}, B_{z_1}^{r_1}, B_{z_2}^{r_2}, \dots, B_{z_n}^{r_n}$, respectively, such that

$$\Phi_i(z_i) = 0, \quad i = -1, 1, \dots, n \tag{3.1.2}$$

$$\lim_{w \to \infty} w\Phi_{-1}(w) = 1, \tag{3.1.3}$$

$$\lim_{w \to z_i} \frac{\Phi_i(w)}{w - z_i} \neq 0, \quad i = 1, \dots, n. \tag{3.1.4}$$

This proposition is exactly what is proven in Proposition 1.3.1 of [H2].

Remark 3.1.2. *In Huang's work, [H2], the outgoing puncture is labelled the 0-th puncture while the incoming punctures are labelled the first through n-th punctures.*

In order to generalize to multiple outgoing punctures, we refer to outgoing punctures with negative indices. For example, in Proposition 3.1.1 the lone outgoing puncture is referred to as z_{-1} *and it's local coordinate chart is referred to as* $(B^{r_{-1}}_{z_{-1}}, \Phi_{-1})$ *whereas these would be referred to as* z_0 *and* $(B^{r_0}_{\infty}, \Phi_0)$*, respectively, in [H2]. Our new notation is required to consider multiple outgoing punctures but also highlights the natural symmetry between incoming and outgoing punctures.*

The fact that every conformal equivalence class of spheres of type $(1, n)$, for $n \in \mathbb{Z}_+$, contains a sphere of the form (3.1.1) allows us to focus solely on the Riemann sphere, $\widehat{\mathbb{C}}$, with tubes and still include every equivalence class. The following proposition goes a step farther and describes the precise amount of information needed to specify a particular equivalence class.

Proposition 3.1.3. *Let*

$$\Sigma_1 = (\widehat{\mathbb{C}}; \infty, z_1, \dots, z_{n-1}, 0; (B^{r_{-1}}_{\infty}, \Phi_{-1}), (B^{r_1}_{z_1}, \Phi_1), \dots, (B^{r_{n-1}}_{z_{n-1}}, \Phi_{n-1}), (B^{r_n}_0, \Phi_n))$$

and

$$\Sigma_2 = (\widehat{\mathbb{C}}; \infty, \zeta_1, \dots, \zeta_{n-1}, 0; (B^{s_{-1}}_{\infty}, \Psi_{-1}), (B^{s_1}_{\zeta_1}, \Psi_1), \dots (B^{s_{n-1}}_{\zeta_{n-1}}, \Psi_{n-1}), (B^{s_{-n}}_0, \Psi_{-n}))$$

be two spheres of type $(1, n)$*, for* $n \in \mathbb{Z}_+$*. Let* $f_{-1}, f_1, \dots, f_n$ *and* $g_{-1}, g_1, \dots, g_n$ *be the series obtained by expanding the analytic functions* $\Phi_{-1}, \Phi_1, \dots, \Phi_n$ *and* $\Psi_{-1}, \Psi_1, \dots, \Psi_n$ *around* $w = \infty, z_1, \dots, z_{n-1}, 0$ *and* $w = \infty, \zeta_1, \dots, \zeta_{n-1}, 0$*, respectively. The worldsheets* Σ_1 *and* Σ_2 *are conformally equivalent if and only if* $z_i = \zeta_i$ *for* $i = 1, \dots, n-1$*, and* $f_i = g_i$ *(as power series), for* $i = -1, 1, \dots, n$*.*

This proposition along with its proof may by found in [H2] (Proposition 1.3.3). To extend the above results to spheres of type $(1, 0)$, we need the following two propositions (Propositions 1.3.4 and 1.3.6 in [H2], respectively).

Proposition 3.1.4. *Any sphere with tubes of type* $(1,0)$ *is conformally equivalent to a sphere of the form*

$$(\widehat{\mathbb{C}}; \infty; (B_\infty^{r_{-1}}, \Phi_{-1}))$$

where $r_1 \in \mathbb{R}_+$ *and* Φ_{-1} *is an analytic function on* $B_\infty^{r_{-1}}$ *that can be expanded as*

$$\Phi_{-1}(w) = \frac{1}{w} + \sum_{j=2}^{\infty} a_j \left(\frac{1}{w}\right)^{j+1} \tag{3.1.5}$$

with each $a_j \in \mathbb{C}$.

Proposition 3.1.5. *Two spheres of the form* $Q_1 = (\widehat{\mathbb{C}}; \infty; (B_\infty^{r_{-1}}, \Phi_{-1}))$ *and* $Q_2 = (\widehat{\mathbb{C}}; \infty; (B_\infty^{s_{-1}}, \Psi_{-1}))$ *are conformally equivalent if and only if* $f_{-1} = g_{-1}$ *(as power series), where* f_{-1} *and* g_{-1} *are the power series expansions of* Φ_{-1} *and* Ψ_{-1}*, respectively, around* $w = \infty$.

These propositions allow us to canonically describe conformal equivalence classes of spheres of type $(1,n)$ for $n \in \mathbb{N}$ as follows:

$$(z_1, \ldots, z_{n-1}; f_{-1}, f_1, \ldots, f_n) \quad \text{if } n > 0 \tag{3.1.6}$$

$$(f_{-1}) \quad \text{if } n = 0 \tag{3.1.7}$$

where $z_1, \ldots, z_{n-1}$ are distinct nonzero complex numbers and $f_{-1}, f_1, \ldots, f_n$ are power series that are convergent in some positive neighborhood, satisfying (3.1.2)-(3.1.4) for $n > 0$ and (3.1.5) for $n = 0$. These tuples represent the conformal equivalence classes containing the worldsheets

$$(\widehat{\mathbb{C}}; \infty, z_1, \ldots, z_{n-1}, 0; (B_\infty^{r_{-1}}, f_{-1}), (B_{z_1}^{r_1}, f_1), \ldots, (B_{z_{n-1}}^{r_{n-1}}, f_{n-1}), (B_0^{r_n}, f_n)) \quad \text{if } n > 0$$

and

$$(\widehat{\mathbb{C}}; \infty; (B_\infty^{r_{-1}}, f_{-1})) \quad \text{if } n = 0,$$

respectively, where $r_{-1}, r_1, \ldots, r_n$ are appropriately chosen radii of convergence so that the corresponding local coordinate maps are convergent within $B_{\infty}^{r_{-1}}$, $B_{z_1}^{r_1}, \ldots,$ $B_{z_{n-1}}^{r_{n-1}}$, $B_0^{r_n}$, respectively, and the local coordinate neighborhoods do not overlap. Notice that the lack of specificity on the choice of the r_i means that we are not choosing a specific canonical representative of each equivalence class in the most specific sense. This subtlety is usually suppressed and the above tuples are referred to as canonical representatives because the germ of each analytic function is the only data affecting which equivalence class a particular sphere with tubes belongs to. We refer to the set of conformal equivalence classes of spheres with incoming tubes as *the moduli space of spheres with incoming tubes*.

3.2 The moduli space of spheres with outgoing tubes

We now turn our attention to spheres with outgoing tubes. The results for spheres of type $(n, 1)$, for $n \in \mathbb{Z}_+$ follow directly from the spheres with incoming tubes case since the proofs of those propositions never refer to the orientations of punctures. In the case of spheres of type $(0, 1)$, however, additional proof is necessary in order to normalize the lone incoming puncture at 0 instead of at ∞ as is the case for spheres of type $(1, 0)$. (We normalize the single incoming puncture in the representative of conformal equivalence classes of spheres of type $(0, 1)$ to be at 0 in order to maintain compatibility with the preexisting structure for GVOAs defined by Huang in [H2].)

Proposition 3.2.1. *Any sphere with tubes of type $(n, 1)$, for $n \in \mathbb{Z}_+$, is conformally equivalent to a sphere of the form*

$$(\widehat{\mathbb{C}}; z_{-n}, \ldots, z_{-1}, z_1; , (B_{z_{-n}}^{r_{-n}}, \Phi_{-n}), \ldots, (B_{z_{-1}}^{r_{-1}}, \Phi_{-1}), (B_{z_1}^{r_1}, \Phi_1))$$

where $z_{-n} = \infty$, $z_1 = 0$, $z_i \in \mathbb{C}^\times$ for $i = -n+1, \ldots, -1$, satisfying $z_i \neq z_j$ for $i \neq j$, $r_i \in \mathbb{R}_+$ for $i = -n, \ldots, -1, 1$ and $\Phi_{-n}, \ldots, \Phi_{-1}, \Phi_1$ are analytic on

$B^{r_{-n}}_{z_{-n}}, \dots, B^{r_{-1}}_{z_{-1}}, B^{r_1}_{z_1}$, *respectively, such that*

$$\Phi_i(z_i) = 0 \quad i = -n, \dots, -1, 1 \tag{3.2.1}$$

$$\lim_{w \to \infty} w\Phi_{-n}(w) = 1, \tag{3.2.2}$$

$$\lim_{w \to z_i} \frac{\Phi_i(w)}{w - z_i} \neq 0, \quad i = -n+1, \dots, -1, 1. \tag{3.2.3}$$

Proof. The proof of Proposition 1.3.1 in [H2] suffices since it never uses the orientation of the punctures. □

As before, this proposition narrows the choices for a canonical representatives in each equivalence class of spheres of type $(n, 1)$ and the following proposition illuminates the exact minimum information required to reference a particular equivalence class.

Proposition 3.2.2. *Let*

$$\Sigma_1 = (\widehat{\mathbb{C}}; \infty, z_{-n+1}, \dots, z_{-1}, 0; (B^{r_{-n}}_{\infty}, \Phi_{-n}), (B^{r_{-n+1}}_{z_{-n+1}}, \Phi_{-n+1}), \dots, (B^{r_{-1}}_{z_{-1}}, \Phi_{-1}), (B^{r_1}_0, \Phi_1))$$

and

$$\Sigma_2 = (\widehat{\mathbb{C}}; \infty, \zeta_{-n+1}, \dots, \zeta_{-1}, 0; (B^{s_{-n}}_{\infty}, \Psi_{-n}), (B^{s_{-n+1}}_{\zeta_{-n+1}}, \Psi_{-n+1}), \dots, (B^{s_{-1}}_{\zeta_{-1}}, \Psi_{-1}), (B^{s_1}_0, \Psi_1))$$

be two spheres of type $(n, 1)$, for $n \in \mathbb{Z}_+$. Let $f_{-n}, \dots, f_{-1}, f_1$ and $g_{-n}, \dots, g_{-1}, g_1$ be the series obtained by expanding the analytic functions $\Phi_{-n}, \dots, \Phi_{-1}, \Phi_1$ and $\Psi_{-n}, \dots, \Psi_{-1}, \Psi_1$ around $w = \infty, z_{-n+1}, \dots, z_{-1}, 0$ and $w = \infty, \zeta_{-n+1}, \dots, \zeta_{-1}, 0$, respectively. The worldsheets Σ_1 and Σ_2 are conformally equivalent if and only if $z_i = \zeta_i$, for $i = -n+1, \dots, -1$ and $f_i = g_i$ (as power series), for $i = -n, \dots, -1, 1$.

Proof. Follows from Proposition 1.3.3 in [H2]. □

These two propositions allow us to choose a canonical representative for each conformal equivalence class of worldsheets of type $(n, 1)$, for $n \in \mathbb{Z}_+$. Here we switch notation slightly for power series centered at points other than 0 and ∞. Power series expanded thus far about 0, or about any nonzero complex number, have been power series in w, while the power series at ∞ had to be expanded in terms of $\frac{1}{w}$. However, for the nonzero complex punctures, expanding in terms of $\frac{1}{w}$ is also valid. In describing representatives for equivalence classes of spheres with outgoing tubes, we will use the latter convention for all outgoing (i.e., nonzero) punctures. Thus a canonical representative will be denoted

$$(\frac{1}{z_{-n+1}}, \ldots, \frac{1}{z_{-1}}; f_{-n}, \ldots, f_{-1}, f_1), \tag{3.2.4}$$

where $z_{-n+1}, \ldots, z_{-1}$ are distinct nonzero complex numbers, $f_{-n} \ldots, f_{-1}$ are power series in $\frac{1}{w}$ centered at $\frac{1}{z_{-n+1}}, \ldots, \frac{1}{z_{-1}}$, respectively, that are convergent in some positive neighborhood, and f_1 is a power series in w centered at 0 that is convergent in some positive neighborhood, which satisfy (3.2.1)-(3.2.3). This canonical representative will represent the equivalence class of worldsheets containing the worldsheet

$$(\widehat{\mathbb{C}}; \infty, z_{-n+1}^{-1}, \ldots, z_{-1}^{-1}, 0; (B_{\infty}^{r_{-n}}, f_{-n}), (B_{z_{-n+1}^{-1}}^{r_{-n+1}}, f_{-n+1}), \ldots, (B_{z_{-1}^{-1}}^{r_{-1}}, f_{-1}), (B_0^{r_1}, f_1)),$$

where $r_{-n}, \ldots, r_{-1}, r_1$ are appropriately chosen radii of convergence so that each local coordinate map is convergent within its corresponding local coordinate neighborhood.

It turns out to be useful to refer to nonzero punctures as $\frac{1}{z}$, first because we will be applying the global transformation $w \mapsto \frac{1}{w}$ to the Riemann spheres used in [H2] in some sense (cf. Section 4.4), second because it makes composing multiple shifts to ∞ clearer, and third because it simplifies the isomorphism between geometric vertex operator coalgebras and vertex operator coalgebras in Chapter 8. Again, notice that to actually choose a true canonical representative of the conformal equivalence class, one would need to fix the r_i's.

We use a similar approach to establish canonical representatives for conformal equivalence classes of worldsheets of type $(0,1)$, however, the proofs do not come directly from the case of spheres with incoming tubes.

Proposition 3.2.3. *Any sphere with tubes of type* $(0,1)$ *is conformally equivalent to a sphere of the form*

$$(\widehat{\mathbb{C}}; 0; (B_0^{r_1}, \Phi_1))$$

where $r_1 \in \mathbb{R}_+$ *and* Φ_1 *is an analytic function on* $B_0^{r_1}$ *that can be expanded as*

$$\Phi_1(w) = w + \sum_{j=2}^{\infty} a_j w^{j+1}, \tag{3.2.5}$$

for $a_j \in \mathbb{C}$.

Proof. Let

$$Q = (S; p_1; (U_1, \Psi_1))$$

be a sphere with tubes of type $(0,1)$. We know that S is complex analytically isomorphic to the sphere $\widehat{\mathbb{C}}$ so let $F : S \to \widehat{\mathbb{C}}$ be such an isomorphism. Let $(\Psi_1 \circ F^{-1})(w)$ be expanded as a power series about $F(p_1)$ so that

$$(\Psi_1 \circ F^{-1})(w) = \sum_{j=0}^{\infty} b_j (w - F(p_1))^{j+1}$$

with each $b_j \in \mathbb{C}$ and $b_0 \neq 0$. Define a transformation $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by

$$T(w) = \frac{b_0(w - F(p_1))}{-(\frac{b_1}{b_0}(w - F(p_1)) + 1}$$

Since $\Psi_1 \circ F^{-1} \circ T^{-1}$ is analytic in some neighborhood of 0, there exists $r_1 \in \mathbf{R}_+$ such that

$$\Psi_1 \circ F^{-1} \circ T^{-1}(w) = w + \sum_{j=2}^{\infty} a_j w^{j+1}$$

in $B_0^{r_1}$ with each $a_j \in \mathbb{C}$. This equation may be verified by taking derivatives of both sides at 0. Thus Q is conformally equivalent to

$$Q = (\widehat{\mathbb{C}}; 0; (B_0^{r_1}, \Psi_1 \circ F^{-1} \circ T^{-1}))$$

via the complex analytic isomorphism $T \circ F$. □

Proceeding in the same way as we did in Proposition 3.1.5 for worldsheets of type $(1, 0)$, we now argue:

Proposition 3.2.4. *Two spheres of the form $Q_1 = (\widehat{\mathbb{C}}; 0; (B_0^{r_1}, \Phi_1))$ and $Q_2 = (\widehat{\mathbb{C}}; 0; (B_0^{s_1}, \Psi_1))$ are conformally equivalent if and only if $f_1 = g_1$ (as power series), where f_1 and g_1 are the power series expansions of Φ_1 and Ψ_1, respectively, around $w = 0$.*

Proof. Let F be the conformal equivalence from Q_1 to Q_2. The latter half of the proposition is equivalent to F being the identity map on $\widehat{\mathbb{C}}$. We know that F is a projective transformation with $F(0) = 0$ so

$$F(w) = \frac{aw}{cw + 1}$$

for some $a \in \mathbb{C}^\times$ and $c \in \mathbb{C}$. We know

$$g_1|_{B_0^r} = f_1 \circ F^{-1}|_{B_0^r},$$

for some $r > 0$, and

$$\lim_{w \to 0} \frac{1}{w} g_1(w) = 1$$

so

$$\lim_{w \to 0} \frac{1}{w} f_1 \left(\frac{aw}{cw + 1} \right) = 1. \tag{3.2.6}$$

Since f_1 is of the form

$$w + \sum_{j=2}^{\infty} a_j w^{j+1},$$

we see from (3.2.6) that $a = 1$. Further, since

$$\lim_{w\to 0} \frac{1}{w^2}(g_1(w) - w) = 0,$$

we have

$$\lim_{w\to 0} \frac{1}{w^2}(f_1\left(\frac{w}{cw+1}\right) - w) = 0,$$

and thus by considering the form of f_1 we see that $c = 0$ making F the identity map. The converse argument is trivial. □

A representative for an equivalence class of spheres of type $(0, 1)$ will be described as

$$(f_1),$$

where f_1 is a power series satisfying (3.2.5) and convergent in some positive neighborhood, which will represent the equivalence class of spheres of type $(1, 0)$ containing

$$(\widehat{\mathbb{C}}; 0; (B_0^{r_1}, f_1(w)))$$

where r_1 is an appropriately chosen radius of convergence. We refer to the set of conformal equivalence classes of spheres with outgoing tubes as *the moduli space of spheres with outgoing tubes.*

Remark 3.2.5. *Although we will be primarily concerned with the moduli space of spheres with incoming tubes and the moduli space of spheres with outgoing tubes separately, the above propositions allow us to describe the general moduli space of spheres of type* (m, n)*, for any* $m, n \in \mathbb{N}$*. The type* $(0, 0)$ *case is a one element set (that is not sewable); the cases of spheres of types* $(1, 0)$ *and* $(0, 1)$ *have been described above; and given* $m, n \in \mathbb{Z}_+$*, the moduli spaces of spheres of type* (m, n) *may be describes as*

$$(\frac{1}{z_{-m+1}}, \ldots, \frac{1}{z_{-1}}, z_1, \ldots, z_{n-1}; f_{-m}, \ldots, f_{-1}, f_1, \ldots, f_n) \tag{3.2.7}$$

by making the trivial generalization to Proposition 3.1.1 or 3.2.1 that the orientations of punctures other than the first and last are irrelevant. Here Equation (3.2.7) is the generalization of Equations (3.1.6) and (3.2.4), and should be understood similarly.

3.3 Sewing moduli spaces of spheres with tubes

Now that we have described the moduli space of spheres with tubes (the moduli space of spheres with incoming tubes and the moduli space of spheres with outgoing tubes being subsets), we need to lift the sewing operation from the sewing of particular spheres to the sewing of elements of the moduli space. Given Σ_1 and Σ_2, two worldsheets such that the j-th outgoing puncture of Σ_2 may be sewn into the i-th incoming puncture of Σ_1 as in Section 2.2, let Σ_3 be the worldsheet resulting from the sewing. We need to be certain that given Σ_1' and Σ_2' in the same conformal equivalence classes as Σ_1 and Σ_2, respectively, that the j-th outgoing puncture of Σ_2' may be sewn into the i-th incoming puncture of Σ_1' and that the resulting sphere, Σ_3', is in the same equivalence class as Σ_3. But given complex analytic isomorphisms $G_1 : \Sigma_1 \to \Sigma_1'$ and $G_2 : \Sigma_2 \to \Sigma_2'$ witnessing conformal equivalence, a radius $r \in \mathbb{R}_+$ that make Σ_1 and Σ_2 sewable is exactly one that makes Σ_1' and Σ_2' sewable. Further, $G_1 \sqcup G_2$ will provide precisely the complex analytic isomorphism needed between Σ_3 and Σ_3'.

When it is the case that two elements of the moduli space of spheres, Q_1 and Q_2, are sewable with the j-th outgoing tube of Σ_2 attaching to the i-th incoming tube of Σ_1, we denote the resulting sewn element of the moduli space by $Q_1 \; {}_i\infty_{-j} Q_2$. Now that we have established sewing on equivalence classes, the main work of this section is to develop this notion in terms of canonical representatives of equivalence classes.

Let

$$Q_1 = (z_{-m+1}^{-1}, \ldots, z_{-1}^{-1}, z_1, \ldots, z_{n-1}; f_{-m}, \ldots, f_{-1}, f_1, \ldots, f_n)$$

$$Q_2 = (\zeta_{-k+1}^{-1}, \ldots, \zeta_{-1}^{-1}, \zeta_1, \ldots, \zeta_{\ell-1}; g_{-k}, \ldots, g_{-1}, g_1, \ldots, g_\ell)$$

be two elements of the moduli space with spheres such that $Q_1 \;{}_i\infty_{-j} Q_2$ exists. Equivalently we may require that given the canonical representatives of Q_1 and Q_2, for $\ell, m \in \mathbb{N}$, $k, n \in \mathbb{Z}_+$ and $1 \leq i \leq n$, $1 \leq j \leq k$, there exist $r > 0$ with $f_i^{-1}(\overline{B}^r)$ and $g_{-j}^{-1}(\overline{B}^{1/r})$ well defined and containing only the punctures z_i and ζ_{-j}^{-1}, respectively (note that we consider $z_n = 0$ and $\zeta_{-k}^{-1} = \infty$). In this case, there are r_1 and r_2 satisfying $0 < r_2 < r < r_1$ such that $f_i^{-1}(\overline{B}^{r_1})$ and $g_{-j}^{-1}(\overline{B}^{1/r_1})$ are well defined and still contain only the punctures z_i and ζ_{-j}^{-1}, respectively. Choose

$$Q_3 = (\tilde{z}_{-m-k+2}^{-1}, \ldots, \tilde{z}_{-1}^{-1}, \tilde{z}_1, \ldots, \tilde{z}_{n+\ell-2}; \tilde{f}_{-m-k+1}, \ldots, \tilde{f}_{-1}, \tilde{f}_1, \ldots, \tilde{f}_{n+\ell-1})$$

so that $Q_1 \;{}_i\infty_{-j} Q_2 = Q_3$. Let F be the unique conformal equivalence from the sewing of the canonical representatives of Q_1 and Q_2 to the canonical representative of Q_3. Since the sewing is in two components, there are also unique maps $F^{(1)} : \widehat{\mathbb{C}} \setminus f_i^{-1}(\overline{B}^{r_2}) \to \widehat{\mathbb{C}}$ and $F^{(2)} : \widehat{\mathbb{C}} \setminus g_{-j}^{-1}(\overline{B}^{1/r_1}) \to \widehat{\mathbb{C}}$ satisfying

$$F^{(1)}(w)\big|_{f_i^{-1}(B^{r_2} \setminus \overline{B}^{r_1})} = F^{(2)}\left(g_{-j}^{-1}\left(\frac{1}{f_i(w)}\right)\right)\bigg|_{f_i^{-1}(B^{r_2} \setminus \overline{B}^{r_1})}, \tag{3.3.1}$$

$$F^{(1)}(\infty) = \infty, \tag{3.3.2}$$

$$F^{(2)}(0) = 0, \tag{3.3.3}$$

$$\lim_{w \to \infty} \frac{F^{(1)}(w)}{w} = 1. \tag{3.3.4}$$

We call (3.3.1) the *sewing equation* and (3.3.2), (3.3.3), (3.3.4) the *normalization conditions*. These are the same equations as developed in Section 1.4 of [H2] but in a slightly more general context. A good deal of work will go into describing the canonical representative of $Q_1 \;{}_i\infty_{-j}\; Q_2$ constructively. We will compute several explicit examples in Section 4.5.

3.4 A generating set for the moduli space of spheres with tubes

In operads, or indeed in nearly any structure with an operation, it is valuable to investigate what set together with the available operations generates all possible elements. That question is answered for the moduli space of spheres with incoming tubes in Proposition 1.3.9 of [H2] as follows.

Proposition 3.4.1. *Any element of the moduli space of spheres of type* $(1, n)$*, for* $n \in \mathbb{N}$*, is generated under sewing by the element* (w^{-1}) *of type* $(1, 0)$*, elements of type* $(1, 1)$*, and elements of type* $(1, 2)$ *which are of the form* $(z; w^{-1}, w - z, w)$*, for* $z \in \mathbb{C}^{\times}$*.*

In the proposition w^{-1}, $w - z$, and w should be understood as power series in w^{-1}, w, and w centered at ∞, z and 0, respectively. We are able to demonstrate a similar generating set for the moduli space of spheres with outgoing tubes (using similar notation for power series).

Proposition 3.4.2. *Any element* Q *of the moduli space of spheres of type* $(n, 1)$*, for* $n \in \mathbb{N}$*, is generated under sewing by the element* (w) *of type* $(0, 1)$*, elements of type* $(1, 1)$*, and elements of type* $(2, 1)$ *which are of the form* $(z^{-1}; w^{-1}, w^{-1} - z, w)$*, for* $z \in \mathbb{C}^{\times}$*.*

Proof. Equivalence classes of type $(0, 1)$, $(1, 1)$ and $(2, 1)$ can easily be obtained from the above generating elements. For elements of $(n, 1)$, with $n > 2$, we will use induction.

It suffices to show that elements of the form

$$Q = (z_{-n+1}^{-1}, \ldots, z_{-1}^{-1}; w^{-1}, w^{-1} - z_{-n+1}, \ldots, w^{-1} - z_{-1}, w)$$

can be generated since sewing appropriate elements of $(1, 1)$ to each puncture gives the appropriate local coordinates. Let γ be a Jordan curve in $\mathbb{C}^{\times} \cup \{\infty\}$ such that ∞ and z_{-n+1}^{-1} are in the interior and $z_{-n+2}^{-1}, \ldots, z_{-1}^{-1}, 0$ are in the exterior. By the

Riemann mapping theorem there is a conformal map, f, from the interior of γ onto the unit disk which maps ∞ to 0.

Let

$$a = \lim_{w \to \infty} w\ f(w) \in \mathbb{C}^{\times}.$$

and define $T(w) = aw$, $I(w) = w^{-1}$ and $S_z(w) = \frac{1}{w} - z$. By induction the two elements

$$Q_1 = (T \circ I \circ f(z_{-n+1}^{-1}); I \circ f^{-1} \circ I \circ T^{-1}, S_{z_{-n+1}} \circ f^{-1} \circ I \circ T^{-1}, T^{-1})$$

$$Q_2 = (T(z_{-n+2}^{-1}), \ldots, T(z_{-1}^{-1}); f \circ T^{-1}, S_{z_{-n+2}} \circ T^{-1}, \ldots, S_{z_{-1}} \circ T^{-1}, T^{-1}),$$

of type $(2, 1)$ and $(n-1, 1)$, respectively, are generated by the required set of elements, with $Q_1\ {}_1\infty_{-n+1} Q_2 = Q$ completing the inductive step. $\square$

3.5 Expansions in terms of infinitesimal local coordinates

Any local coordinate map at 0 may be expressed uniquely as

$$\exp\left(\sum_{j \in \mathbb{Z}_+} A_j w^{j+1} \frac{d}{dw}\right) a_0^{w \frac{d}{dw}} w \tag{3.5.1}$$

where the $A_j \in \mathbb{C}$ and $a_0 \in \mathbb{C}^{\times}$ (cf. Proposition 2.1.1 in [H2] and the following discussion).

Any local coordinate map vanishing at ∞, on the other hand, may be written as

$$\exp\left(-\sum_{j \in \mathbb{Z}_+} A_j w^{-j+1} \frac{d}{dw}\right) a_0^{-w \frac{d}{dw}} \frac{1}{w} \tag{3.5.2}$$

where the $A_j \in \mathbb{C}$ and $a_0 \in \mathbb{C}^{\times}$ (Proposition 2.1.16 in [H2]).

Remark 3.5.1. *In Huang's treatment of these moduli spaces as well as in this work, each canonical representative has local coordinate map of puncture at ∞ normalized so that $a_0 = 1$. This is a direct consequence of the conditions (3.1.3) and (3.1.5) in the moduli space of spheres with incoming tubes and the condition (3.2.2) in the* *moduli space of spheres with outgoing tubes. The decision here to use the third degree of freedom in choosing a canonical representative for the purpose of normalizing local coordinate maps at ∞, allows for compatibility with Huang's work [H2] in terms of Virasoro relations and in terms of spheres of type* $(1,1)$.

Any local coordinate map vanishing at a nonzero complex number, z, may be written as a local coordinate map at 0 composed with a shift of z to 0:

$$\exp\left(\sum_{j\in\mathbb{Z}_+} A_j x^{j+1}\frac{d}{dx}\right) a_0^{x\frac{d}{dx}} x\bigg|_{x=w-z},$$

or as a local coordinate map at ∞ composed with a shift of z to ∞:

$$\exp\left(-\sum_{j\in\mathbb{Z}_+} A_j x^{-j+1}\frac{d}{dx}\right) a_0^{-x\frac{d}{dx}} \frac{1}{x}\bigg|_{x=(w^{-1}-z^{-1})^{-1}}.$$

We will use the expansion at 0 for incoming tubes (as in [H2]) and the expansion at ∞ for outgoing tubes. Let the sequence $A = \{A_i\}_{i\in\mathbb{Z}_+}$ denote the higher order coefficients in a given exponential expansion in terms of the infinitesimal local coordinates, $x^{j+1}\frac{d}{dx}$, and let the notation $(a_0, (A_1, A_2, \ldots))$ or (a_0, A) record all the coefficients in a given expansion. We will also use the notation $\mathbf{0}$ for the sequence of all zeros and $A(a)$ for $\{a^i A_i\}_{i\in\mathbb{Z}_+}$. When we are dealing with a sequence of formal variables, we will denote the sequence $(\alpha_0, (\mathcal{A}_1, \mathcal{A}_2, \ldots))$ or $(\alpha_0, \mathcal{A})$ following [H2] and [B].

Using this notation, the canonical representative for an equivalence class of genus-zero worldsheets of type $(1,n)$ can be expressed as

$$(z_1, \ldots, z_{-n+1}; A^{(-1)}, (a_0^{(1)}, A^{(1)}), \ldots, (a_0^{(n)}, A^{(n)})), \tag{3.5.3}$$

and the canonical representative for an equivalence class of type $(n,1)$ can be expressed as

$$(\frac{1}{z_{-n+1}}, \ldots, \frac{1}{z_{-1}}; A^{(-n)}, (a_0^{(-n+1)}, A^{(-n+1)}), \ldots, (a_0^{(-1)}, A^{(-1)}), (a_0^{(1)}, A^{(1)})), \tag{3.5.4}$$

with $n \in \mathbb{Z}_+$, the z_i's distinct nonzero complex numbers, $(a_0^i, A^{(i)})$ recording the local coordinate map for the puncture of corresponding index, and for the first sequence in each list, $A^{(-1)}$ and $A^{(-n)}$ respectively, recording the higher order coefficients for the exponential expansion of the local coordinate map at ∞. As mentioned before, these local coordinate maps will always be normalized to have $a_0^{(-1)} = 1$ and $a_0^{(-n)} = 1$, respectively. Notice that for spheres with outgoing tubes the i-th outgoing puncture is at $1/z_{-i}$ so that the associated local coordinate map will be

$$\exp\left(-\sum_{j\in\mathbb{Z}_+} A_j^{(i)} x^{-j+1}\frac{d}{dx}\right)(a_0^{(i)})^{-x\frac{d}{dx}}\frac{1}{x}\Bigg|_{x=(w^{-1}-z_{-i})^{-1}}.$$

Exponential notation also allows us to describe a canonical representative of type $(1,0)$ as

$$(A^{(-1)}) \tag{3.5.5}$$

where $A^{(-1)}$ records the coefficients of the exponential expansion at ∞, and to describe a canonical representative of type $(0,1)$ as

$$((1, A^{(1)})), \tag{3.5.6}$$

where $(1, A^{(1)})$ records the local coordinate map for the puncture at 0. We see from Proposition 3.1.4 that in the type $(1,0)$ case not only must $a_0^{(-1)} = 1$ but $A_1^{(-1)} = 0$ for all representatives. Also, Proposition 3.2.3 implies that in the type $(0,1)$ case $a_0^{(1)} = 1$ and $A_1^{(1)} = 0$.

3.6 The notions of K and K^*

We now formalize the descriptions of the moduli space of spheres with one outgoing and n ordered incoming tubes. This language will be quite valuable in discussing the operadic interpretation of the moduli space. For $n \in \mathbb{Z}_+$, let

$$M^{n-1} = \{(z_1, z_2, \ldots z_{n-1}) | z_i \in \mathbb{C}^\times, z_i \neq z_j \text{ for } i \neq j\}$$

Note that for $n = 1$, M^0 has exactly one element. We say that the series $\exp\left(\sum_{j\in\mathbb{Z}_+} A_j w^{j+1}\frac{d}{dw}\right) w$ is absolutely convergent in a neighborhood of 0 precisely if its expansion as powers of w is absolutely convergent in a neighborhood of 0. We then define

$$H = \Bigg\{ A = \{A_j\}_{j\in\mathbb{Z}_+} \in \mathbb{C}^\infty \mid \exp\left(\sum_{j\in\mathbb{Z}_+} A_j w^{j+1}\frac{d}{dw}\right) w \text{ is absolutely convergent in some neighborhood of } 0 \Bigg\}. \tag{3.6.1}$$

Proposition 3.6.1. *The moduli space of spheres with tubes of type* $(1, n)$*, for* $n \in \mathbb{Z}_+$*, can be identified with the set*

$$K(n) = M^{n-1} \times H \times (\mathbb{C}^\times \times H)^n,$$

and the moduli space of spheres with tubes of type $(1, 0)$ *can be identified with the set*

$$K(0) = \{A \in H | A_1 = 0\}.$$

This is simply a restatement of Propositions 3.1.1 and 3.1.2 in [H2] but is the obvious conclusion of Equations (3.5.3) and (3.5.5). The natural object of consideration is then

$$K = \coprod_{n\in\mathbb{N}} K(n), \tag{3.6.2}$$

which we will call the *moduli space of spheres with incoming tubes.*

Having established precisely what we mean by the moduli space of spheres with one incoming, n ordered outgoing punctures and corresponding vanishing local coordinates, we move to the moduli space of spheres with n ordered incoming punctures, one outgoing puncture and corresponding vanishing local coordinates. Via Equations (3.5.4) and (3.5.6), we now establish a uniform description that will essentially be identified with each element of the moduli space.

Proposition 3.6.2. *Let* $\widetilde{M}^{n-1} = \{(\frac{1}{z_{-n+1}}, \ldots \frac{1}{z_{-1}}) | z_i \in \mathbb{C}, z_i \neq z_j$ *for* $i \neq j\}$ *and again let* $H = \{A = \{Aj\}_{j \in \mathbb{Z}_+} | \exp\left(\sum_{j \in \mathbb{Z}_+} A_j w^{j+1} \frac{d}{dw}\right) w$ *is absolutely convergent in some neighborhood of 0* $\}$. *Then the moduli space of spheres with tubes of type* $(n, 1)$, *for* $n \in \mathbb{Z}_+$, *can be identified with the set*

$$K^*(n) = \widetilde{M}^{n-1} \times H \times (\mathbb{C}^\times \times H)^n, \tag{3.6.3}$$

and the moduli space of spheres with tubes of type $(0, 1)$ *can be identified with the set*

$$K^*(0) = \{A \in H | A_1 = 0\}.$$

As in the case of K, we define

$$K^* = \coprod_{n \in \mathbb{N}} K^*(n)$$

to be the *moduli space of spheres with outgoing tubes.*

CHAPTER 4

OPERADIC INTERPRETATIONS OF THE MODULI SPACE OF SPHERES WITH TUBES

Geometric VOAs as well as GVOCs may be understood in terms of operads, a structure first appearing in [St1] and [St2] and first formally defined in [M]. Roughly speaking, operads are $\mathbb{N}$-graded sets of geometric or algebraic objects with a compatible composition operation between elements. Given a morphism from a geometrically defined operad to an algebraically defined operad, one obtains various n-ary operations, induced by the geometry, on the algebraic structure. Section 4.1 will give the basic definitions and provide motivating examples, by which unital associative algebras over $\mathbb{C}$ may be defined. In Section 4.2 we review the operadic interpretation of the moduli space of spheres with incoming tubes and in Section 4.3 we do the same for spheres with outgoing tubes. Finally in Section 4.4 we describe a natural morphism of partial operads between the partial operad described in Section 4.2 and the one in Section 4.3.

4.1 Basic definitions and motivating examples

Definition 4.1.1. *(cf. [M], [HL1]) An* operad C *consists of a family of sets* $C(j)$, $j \in \mathbb{N}$, *together with* (abstract) substitution maps γ *for each* $k, j_1, \dots, j_k \in \mathbb{N}$,

$$\begin{aligned}\gamma : C(k) \times C(j_1) \times \cdots \times C(j_k) &\to C(j_1 + \cdots + j_k)\\ (c; d_1, \dots, d_k) &\mapsto \gamma(c; d_1, \dots, d_k),\end{aligned}$$

an identity element $I \in C(1)$ *and a (left) action of the symmetric group* S_j *on* $C(j)$, $j \in \mathbb{Z}_+$, *satisfying the following axioms:*

(i) Operad-associativity*: For any* $k \in \mathbb{N}$, $j_s, i_t \in \mathbb{N}$, $c \in C(k)$, $d_s \in C(j_s)$ *and* $e_t \in C(i_t)$ *(with* $s = 1, \dots, k$, $t = 1, \dots, j_1 + \ldots + j_k$*),*

$$\gamma(\gamma(c; d_1, \dots, d_k); e_1, \dots, e_{j_1+\cdots+j_k}) = \gamma(c; f_1, \dots, f_k),$$

where

$$f_s = \gamma(d_s; e_{j_1+\cdots+j_{s-1}+1}, \dots, e_{j_1+\cdots+j_s}).$$

(ii) For any $k \in \mathbb{N}$, $c \in C(k)$,

$$\gamma(I; c) = c,$$

$$\gamma(c; I, \dots, I) = c.$$

(iii) For any $k, j_s \in \mathbb{N}$, $c \in C(k)$, $d_s \in C(j_s)$, $\sigma \in S_k$ *and* $\tau_s \in S_{j_s}$ *(with* $s = 1, \dots, k$*),*

$$\gamma(\sigma(c); d_1, \dots, d_k) = \sigma(j_1, \dots, j_k)(\gamma(c; d_{\sigma(1)}, \dots, d_{\sigma(k)})),$$

$$\gamma(c; \tau_1(d_1), \dots, \tau_k(d_k)) = (\tau_1 \oplus \cdots \oplus \tau_k)(\gamma(c; d_1, \dots, d_k)),$$

where $\sigma(j_1, \dots, j_k)$ *denotes the permutation of* $j = j_1 + \ldots + j_k$ *letters which permutes the* k *blocks of letters, determined by the given partition of* j*, in the same way* σ *permutes* k *letters, and* $(\tau_1 \oplus \cdots \oplus \tau_k)$ *denotes the image of* $(\tau_1, \cdots, \tau_k)$ *under the inclusion of* $S_{j_1} \times \cdots \times S_{j_k}$ *into* S_j*.*

We will cover three motivational examples of operads. The first will be a geometric example.

Example 4.1.2. *(cf. [HL1] Example 15) Let $\hat{C}(n)$, for $n \in \mathbb{N}$, be the set of copies of S^1 lying canonically in the complex plane with one distinguished negatively oriented point, or puncture, at the complex number i and n distinct ordered positively oriented points also lying in S^1. Let $C(n)$, for $n \in \mathbb{N}$, be the set of equivalence classes of $\hat{C}(n)$ where two elements C_1 and C_2 are considered equivalent if and only if there is an orientation preserving reparameterization of S^1 mapping i to i and ordered positively oriented punctures to ordered positively oriented punctures in order. (It is not difficult to generalize this description to the equivalence classes, under orientation preserving homeomorphisms lying in the plane, of one dimensional manifolds with punctures. It is also possible to reduce this operad to $C(n) = S_n$ since all that is retained in the equivalence classes is the order of the incoming punctures (cf. [MSS] Proposition 1.10), but then the geometric motivation would be lost.) The composition operation, $\gamma : C(k) \times C(j_1) \times \cdots \times C(j_k) \to C(j_1 + \cdots + j_k)$, for $k, j_1, \ldots, j_k \in \mathbb{N}$ attaches to each incoming puncture of the first circle, one of the remaining k circles in order. Explicitly, given $c \in C(k)$, $d_{j_s} \in C(j_s)$, choose a representative $\hat{c}, \hat{d}_{j_1}, \ldots, \hat{d}_{j_k}$ of each c, d_{j_s}, respectively. For $s = 1, \ldots, k$ choose $\theta_s \in (\pi/2, 5\pi/2)$ such that $e^{i\theta_s}$ is the s-th incoming puncture of $\hat{c}$. Let $\theta_0 = \pi/2$ so that $e^{i\theta_0}$ is the outgoing puncture. Let $r = min\{|\theta_s - \theta_t| : s \neq t\}$. Similarly, for $s = 1, \ldots, k$, $t = 1, \ldots, j_s$, choose $\phi_{s,t} \in (\pi/2, 5\pi/2)$ such that $e^{i\phi_{s,t}}$ is the t-th incoming puncture of $\hat{d}_s$. Then $\gamma(c; d_{j_1}, \ldots, d_{j_k})$ is the equivalence class containing the sphere with one outgoing puncture at i and incoming punctures at*

$$e^{i(\theta_1 - \frac{5r}{4} + \frac{\phi_{1,1} r}{2\pi})}, \ldots, e^{i(\theta_1 - \frac{5r}{4} + \frac{\phi_{1,j_1} r}{2\pi})}, e^{i(\theta_2 - \frac{5r}{4} + \frac{\phi_{2,1} r}{2\pi})}, \ldots, e^{i(\theta_2 - \frac{5r}{4} + \frac{\phi_{2,j_2} r}{2\pi})},$$
$$\ldots, e^{i(\theta_k - \frac{5r}{4} + \frac{\phi_{k,1} r}{2\pi})}, \ldots, e^{i(\theta_k - \frac{5r}{4} + \frac{\phi_{k,j_k} r}{2\pi})}$$

in that order. The unit of this operad is the lone equivalence class of $C(1)$ which has one incoming and one outgoing puncture. For each $n \in \mathbb{N}$, an element $\sigma \in S_n$

of the symmetric group acts on $c \in C(n)$ by reordering the punctures, making the t-th puncture the $\sigma(t)$-th puncture. Verifying the requirements of an operad is not difficult. This operad is denoted $\mathcal{C}irc$.

The operad $\mathcal{C}irc$ is perhaps the best introductory operad and may be interpreted geometrically as loops of string with distinguished points. Another natural example is one coming out of finite-dimensional vector spaces over a field $\mathbf{k}$ (usually taken to be $\mathbb{C}$).

Example 4.1.3. *(cf. [MSS] Definition 1.7) Let V be a finite-dimensional vector space over a field $\mathbf{k}$ and, for each $n \in \mathbb{N}$, let $\mathcal{E}nd_V(n) = Hom_{\mathbf{k}}(V^{\otimes n}, V)$. Define a composition map*

$$\gamma : \mathcal{E}nd_V(k) \times \mathcal{E}nd_V(j_1) \times \cdots \times \mathcal{E}nd_V(j_k) \to \mathcal{E}nd_V(j_1 + \cdots + j_k)$$

by

$$\gamma(f; g_1, \ldots, g_k) = f \circ (g_1 \otimes \cdots \otimes g_k).$$

The unit is the identity map $Id_V : V \to V$ and the left action of S_k on $\mathcal{E}nd_V(k)$ is given by $\sigma(f(x_1, \ldots, x_k)) = f(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$. This operad is denoted either $\mathbf{k}-Vec$ or $\mathcal{E}nd(V)$.

Our third basic example will reverse the arrows of $\mathcal{E}nd(V)$.

Example 4.1.4. *(cf. [MSS] Definition 1.9) Let V be a finite-dimensional vector space over a field $\mathbf{k}$ and, for each $n \in \mathbb{N}$, let $\mathcal{C}o\mathcal{E}nd_V(n) = Hom_{\mathbf{k}}(V, V^{\otimes n})$. Define a composition map*

$$\gamma : \mathcal{C}o\mathcal{E}nd_V(k) \times \mathcal{C}o\mathcal{E}nd_V(j_1) \times \cdots \times \mathcal{C}o\mathcal{E}nd_V(j_k) \to \mathcal{C}o\mathcal{E}nd_V(j_1 + \cdots + j_k)$$

by

$$\gamma(f; g_1, \ldots, g_k) = (g_1 \otimes \cdots \otimes g_k) \circ f.$$

The unit is still the identity map $Id_V : V \to V$ and the left action of S_k on $\mathcal{C}o\mathcal{E}nd_V(k)$ is given by $\sigma(f(x)) = (\sigma \circ f)(x)$ where the latter σ is interpreted to be the map from $v^{\otimes k}$ to $V^{\otimes k}$ sending the t-th input to the $\sigma(t)$-th output. This operad is denoted either $\mathbf{k} - Vec^*$ or $\mathcal{C}o\mathcal{E}nd(V)$. It is also relevant to note that $\mathcal{C}o\mathcal{E}nd(V) \cong \mathcal{E}nd(V^*)$ as operads.

We now investigate the structure that operads induce on multiplication maps by studying morphisms of operads. A *morphism* (cf. [HL1]) $\Psi : C \to D$ of operads C and D is a sequence of S_k-invariant maps $\Psi_k : C(k) \to D(k)$ such that $\Psi_1(I_C) = I_D$ and the following diagram commutes

$$\begin{array}{ccc} C(k) \times C(j_1) \times \cdots \times C(j_k) & \xrightarrow{\gamma_C} & C(j_1 + \cdots + j_k) \\ {\scriptstyle \Psi_k \times \Psi_{j_1} \times \cdots \times \Psi_{j_k}} \downarrow & & \downarrow {\scriptstyle \Psi_{j_1 + \cdots + j_k}} \\ D(k) \times D(j_1) \times \cdots \times D(j_k) & \xrightarrow{\gamma_D} & D(j_1 + \cdots + j_k). \end{array}$$

Given a finite dimensional vector space V, a morphism $\Psi : C \to \mathcal{E}nd(V)$ of operads C and $\mathcal{E}nd(V)$ is said to be a *C-algebra structure* on V and a morphism $\Psi : C \to \mathcal{C}o\mathcal{E}nd(V)$ of operads C and $\mathcal{C}o\mathcal{E}nd(V)$ is said to be a *C-coalgebra structure* on V.

Remark 4.1.5. *It is often useful to reformulate the composition operations in an operad by introducing a binary operation*

$$\circ_i : C(j) \times C(k) \to C(j + k - 1)$$
$$(c, d) \mapsto \gamma(c; \underbrace{I, \ldots, I}_{i-1}, d, \underbrace{I, \ldots, I}_{j-i}).$$

(cf. [H2] Definition C.3.1) Describing $\circ_i$ for all $j \in Z_+$, $k \in \mathbb{N}$, $1 \leq i \leq j$ is equivalent to defining γ since

$$\gamma(c, d_1, \ldots, d_j) = ((\cdots (c \circ_j d_j) \circ_{j-1} d_{j-1}) \circ_{j-2} \cdots \circ_1 d_1).$$

When an operad is formulated purely in terms of the $\circ_i$ operator it is sometimes referred to as an 'o'-sub-'i' operad. *This description of an operad is particularly useful* *when composition of elements is complicated because it only requires the description of the composition of two elements instead of arbitrarily many.*

Example 4.1.6. *We first investigate the family of morphisms $\nu : \mathcal{C}irc \to \mathcal{E}nd(V)$, fixing a finite dimensional vector space V. The choice of a morphism ν is equivalent to the choice of a unital associative algebra structure on V. Specifically, multiplication on V is determined by the binary map $m = \nu_2(Y) \in Hom_{\mathbf{K}}(V \otimes V, V)$, where $Y \in C(2)$ is the element containing the circle with one outgoing puncture at i and two incoming punctures at -1 and 1 in order, and the multiplicative unit of V is determined by $1_V = \nu_0(\mathbf{1}) \in Hom_{\mathbf{K}}(\mathbb{C}, V) = V$, where $\mathbf{1}$ is the single element in $C(0)$. It is clear that m respects the vector space structure. To see that m is associative, first notice that $Y \circ_1 Y = Y \circ_2 Y$. (An element with such a property is said to be associative.) Then observe that*

$$\begin{aligned}
m(m(a,b),c) &= (m \circ_1 m)(a,b,c) \\
&= (\nu_2(Y) \circ_1 \nu_2(Y))(a,b,c) \\
&= (\nu_3(Y \circ_1 Y))(a,b,c) \\
&= (\nu_3(Y \circ_2 Y))(a,b,c) \\
&= (\nu_2(Y) \circ_2 \nu_2(Y))(a,b,c) \\
&= m(a, m(b,c)).
\end{aligned}$$

Observing that 1_V acts as a unit also uses the functoriality of ν. Specifically,

$$\begin{aligned}
m(a, 1_V) &= (m \circ_2 1_V)(a) \\
&= (\nu_2(Y) \circ_2 \nu_0(\mathbf{1}))(a) \\
&= \nu_1(I_{\mathcal{C}irc})(a) \\
&= Id_V(a) \\
&= a
\end{aligned}$$

and similarly for left unicity. Since every element of $\mathcal{C}irc$ may be generated by Y, $\mathbf{1}$, sewings and the symmetric group, the choosing of a unital associative algebra *structure is precisely the choosing of an operad morphism* ν*. One way of viewing this is that the geometry of* $\mathcal{C}irc$ *encodes exactly the information necessary to describe an associative multiplication with unit on vector spaces.*

In a sense, $\mathcal{C}irc$ is the one-dimensional analog of worldsheets with one outgoing and n ordered incoming punctures, and a $\mathcal{C}irc$-algebra structure on V is the analog of an algebra over the (partial) operad of worldsheets with incoming tubes. We now motivate the description of a comultiplicative structure induced by worldsheets with outgoing tubes by describing the comultiplicative structure induced by a $\mathcal{C}irc$-coalgebra structure.

Example 4.1.7. *Fix a finite dimensional vector space* V *and let* ν *be a morphism from* $\mathcal{C}irc$ *to* $\mathcal{C}o\mathcal{E}nd(V)$*. This is equivalent to a counital coassociative coalgebra structure on* V *via the comultiplication* $\Delta = \nu_2(Y) \in Hom_{\mathbf{K}}(V, V \otimes V)$*, where* $Y \in C(2)$ *is as above, and the counit of* V *is determined by* $c_V = \nu_0(\mathbf{1}) \in Hom_{\mathbf{K}}(V, \mathbb{C})$*, where, as before,* $\mathbf{1}$ *is the single element in* $C(0)$*. It is clear that* Δ *respects the module structure and is distributive with respect to the vector space structure as in Example 4.1.6. Associativity is also as in Example 4.1.6:*

$$\begin{aligned}
(\Delta \otimes Id_V)\Delta &= \Delta \circ_1 \Delta \\
&= \nu_2(Y) \circ_1 \nu_2(Y) \\
&= \nu_3(Y \circ_1 Y) \\
&= \nu_3(Y \circ_2 Y) \\
&= \nu_2(Y) \circ_2 \nu_2(Y) \\
&= (Id_V \otimes \Delta)\Delta.
\end{aligned}$$

Finally, for the left counit we see that

$$
\begin{aligned}
(c \otimes Id_V)\Delta &= \Delta \circ_1 c \\
&= \nu_2(Y) \circ_1 \nu_0(\mathbf{1}) \\
&= \nu_1(I_{\mathcal{C}irc}) \\
&= Id_V,
\end{aligned}
$$

and similarly for the right counit.

4.2 The partial operad structure on K, the moduli space of spheres with incoming tubes

Definition 4.2.1. *In the definition above of operads, if the substitution maps γ (or equivalently $\circ_i$) are only partially defined, i.e. γ maps a subset of $C(k) \times C(j_1) \times \cdots \times C(j_k)$ to $C(j_1 + \cdots + j_k)$, and all formulas hold whenever both sides exist, we call the above structure a* partial operad*. If, in addition, a family of sets $C(j)$, for $j \in \mathbb{N}$, fails to satisfy operad-associativity under composition, we refer to the structure as a* partial pseudo-operad.

For partial operads C and D, a morphism from C to D must satisfy the conditions for a morphism of operads and the additional condition that the domains of the substitution maps for C are mapped into the domains for the substitution maps for D. Morphisms of partial pseudo-operads are defined in the same way as for partial operads.

Recall our description of the graded object K in (3.6.2), the moduli space of incoming tubes with spheres. Under the sewing operation (which corresponds to the '$\circ_i$' notation of operads), K is a partial operad ([HL1] Section 5). The action of $\sigma \in S_n$ on a genus-zero Riemann surface with one outgoing and n ordered incoming punctures is given by reordering the punctures and making the i-th incoming puncture the $\sigma(i)$-th incoming puncture for each $i = 1, \ldots, n$. This action is invariant under conformal equivalence and, hence, induces an action of σ on the moduli space of spheres of type $(1, n)$, that is $K(n)$. That action will not be made explicit here. It is, however, made explicit on page 67 of [H2] and the preceding discussion there.

4.3 The partial operad structure on, K^*, the moduli space of spheres with outgoing tubes

Similar to our interpretation of K as a partial operad, we may also consider K^*, the moduli space of spheres with outgoing tubes, as an operad (cf. (3.6.3)). Again, it is the sewing operation that gives K^* the structure of a partial operad. The action of $\sigma \in S_n$ on $K^*(n)$ is the same as that of K, induced from making the i-th outgoing puncture the $\sigma(i)$-th outgoing puncture for each $i = 1, \ldots, n$ on any representative of the equivalence class acted upon. This action is invariant under conformal equivalence and, hence, induces an action of σ on $K^*(n)$. Since K^* will be of major importance to us, we will also explicitly describe that action of S_n on $K^*(n)$.

On each element, $Q \in K^*(n)$, for $n \in Z_+$, the natural left action of the n^{th} symmetric group, S_n, is given by permuting the ordering on the n outgoing punctures and renormalizing. For example, given $(z_{-2}^{-1}, z_{-1}^{-1}; \mathbf{0}, (1, \mathbf{0}), (1, \mathbf{0})) \in K^*(3)$ the action of $\sigma = (1\ 2\ 3) \in S_3$ yields the equivalence class containing the Riemann sphere with trivial local coordinates, one incoming puncture at 0 and three outgoing punctures in the order $\infty, z_{-1}^{-1}, z_{-2}^{-1}$. Notice that this is not, however, a canonical representative in the moduli space, since the last outgoing puncture is not at ∞.

In general, S_n is generated by the transpositions $(i\ i+1)$ for $i = 1, 2, \ldots, n-1$.

Considering the action of the transposition $\sigma = (i\ i+1)$ on $K^*(n)$, if $i < n-1$,

$$
\begin{aligned}
&\sigma(z_{-n+1}^{-1}, \dots, z_{-1}^{-1}; A^{(-n)}, (a_0^{(-n+1)}, A^{(-n+1)}), \dots, (a_0^{(-1)}, A^{(-1)}), (a_0^{(1)}, A^{(1)})) \\
&= (z_{-n+1}^{-1}, \dots, z_{-i-2}^{-1}, z_{-i}^{-1}, z_{-i-1}^{-1}, z_{-i+1}^{-1}, \dots, z_{-1}^{-1}; A^{(-n)}, (a_0^{(-n+1)}, A^{(-n+1)}), \dots, \\
&(a_0^{(-i-2)}, A^{(-i-2)}), (a_0^{(-i)}, A^{(-i)}), (a_0^{(-i-1)}, A^{(-i-1)}), (a_0^{(-i+1)}, A^{(-i+1)}), \\
&\dots, (a_0^{(-1)}, A^{(-1)}), (a_0^{(1)}, A^{(1)})). \qquad (4.3.1)
\end{aligned}
$$

If $i = n-1$, however, the local coordinates of the canonical representative will be changed because of the shift needed to move the last outgoing puncture to ∞ and the need to trivialize the a_0 term in the local coordinate vanishing at that puncture. (The transformation map is $w \to e^{z_{-n+1}w^2\frac{d}{dw}}(a_0^{(-n+1)})^{-w\frac{d}{dw}}w.$) Thus

$$
\begin{aligned}
&\sigma(z_{-n+1}^{-1}, \dots, z_{-1}^{-1}; A^{(-n)}, (a_0^{(-n+1)}, A^{(-n+1)}), \dots, (a_0^{(-1)}, A^{(-1)}), (a_0^{(1)}, A^{(1)})) \\
&= \Big((-a_0^{(-n+1)} z_{-n+1})^{-1}, (a_0^{(-n+1)}(z_{-n+2} - z_{-n+1}))^{-1}, \dots, \\
&(a_0^{(-n+1)}(z_{-1} - z_{-n+1}))^{-1}; A^{(-n+1)}((a_0^{(-n+1)})^{-1}), \\
&\left(\frac{1}{a_0^{(-n+1)}}, A^{(-n)}((a_0^{(-n+1)})^{-1}) \right), \left(\frac{a_0^{(-n+2)}}{a_0^{(-n+1)}}, A^{(-n+2)}((a_0^{(-n+1)})^{-1}) \right), \\
&\dots, \left(\frac{a_0^{(-1)}}{a_0^{(-n+1)}}, A^{(-1)}((a_0^{(-n+1)})^{-1}) \right), \left(a_0^{(1)} a_0^{(-n+1)}, \tilde{A} \right) \Big) \qquad (4.3.2)
\end{aligned}
$$

where $\tilde{A} = \tilde{A}_{k \in \mathbb{Z}_+}$ is defined by

$$
e^{\sum_{k\in\mathbb{Z}_+} \tilde{A}_k w^{k+1}\frac{d}{dw}} = e^{-a_0^{(-n+1)} z_{-n+1} w^2 \frac{d}{dw}} e^{\sum_{k\in\mathbb{Z}_+} (a_0^{(-n+1)})^k A_k^{(1)} w^{k+1}\frac{d}{dw}}
$$

One might fairly ask why we make this calculation explicit. One reason is that this formula reveals the origins of skew-symmetry, Equation (7.2.8), which is a consequence of the definition of vertex operator coalgebra. Another reason, found in Section 8.2, is that the explicit calculation is necessary for the proof of the "permutation axiom" of a geometric vertex operator coalgebra from the axioms of a vertex operator coalgebra.

The next two propositions follow directly from observations on spheres with tubes and the fact that sewing and permutation are invariant under conformal equivalence. (They may also be thought of as extensions of the results of Huang on K ([H2]) to K^* by observing that orientation of punctures is not at issue.)

Proposition 4.3.1. *Let $\ell, m \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ such that $m + n > 1$. Choose $Q_1 \in K^*(\ell)$, $Q_2 \in K^*(m)$, $Q_3 \in K^*(n)$ and integers $1 \le i \le m + n - 1$ and $1 \le j \le n$. Then $Q_1 \ {}_1\infty_{-i} \ (Q_2 \ {}_1\infty_{-j} Q_3)$ exists if and only if one of the following three holds:*

(1) $i < j$, $Q_2 \ {}_1\infty_{-j-\ell+1} \ (Q_1 \ {}_1\infty_{-i} Q_3)$ exists and

$$Q_1 \ {}_1\infty_{-i} \ (Q_2 \ {}_1\infty_{-j} Q_3) = Q_2 \ {}_1\infty_{-j-\ell+1} \ (Q_1 \ {}_1\infty_{-i} Q_3);$$

(2) $j \le i < j + m$, $(Q_1 \ {}_1\infty_{-i+j-1} \ Q_2) \ {}_1\infty_{-j} Q_3$ exists and

$$Q_1 \ {}_1\infty_{-i} \ (Q_2 \ {}_1\infty_{-j} Q_3) = (Q_1 \ {}_1\infty_{-i+j-1} \ Q_2) \ {}_1\infty_{-j} Q_3;$$

(3) $i \ge j + m$, $Q_2 \ {}_1\infty_{-j} \ (Q_1 \ {}_1\infty_{-i+m-1} Q_3)$ exists and

$$Q_1 \ {}_1\infty_{-i} \ (Q_2 \ {}_1\infty_{-j} Q_3) = Q_2 \ {}_1\infty_{-j} \ (Q_1 \ {}_1\infty_{-i+m-1} Q_3).$$

Proposition 4.3.2. *Let $Q_1, Q_2 \in K^*(2)$, $\sigma = (1 \ 2) \in S_2$, and $\tau = (1 \ 3 \ 2) \in S_3$. Then $Q_1 \ {}_1\infty_{-1} \ Q_2$ exists if and only if $\tau(Q_1 \ {}_1\infty_{-2} \ (\sigma \ Q_2))$ also exists. If this is the case,*

$$Q_1 \ {}_1\infty_{-1} \ Q_2 = \tau(Q_1 \ {}_1\infty_{-2} \ (\sigma \ Q_2)).$$

Proposition 4.3.2 may be extended to all of the symmetric groups and spheres with outgoing tubes since every symmetric group is generated by transpositions and every sphere with outgoing tubes is generated as in Proposition 3.4.2. For $Q_1 \in K^*(m)$, $Q_2 \in K^*(n)$, $1 \le i \le n$ so that $Q_1 \ {}_1\infty_{-i} \ Q_2$ exists, we have

$$Q_1 \ {}_1\infty_{-i} \ Q_2 = \tau(Q_1 \ {}_1\infty_{-n} \ (\sigma \ Q_2)),$$

where $\sigma = (i \ n)$ and $\tau = (i \ n+1 \ i+1 \ i+2 \ldots n)$.

4.4 A partial operad isomorphism from K to K^*

It is significant to note that K and K^* are isomorphic as partial operads. The isomorphism is given by the global transformation $I : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} : w \mapsto \frac{1}{w}$ on the underlying Riemann spheres of the canonical representatives of K and K^*; in addition the isomorphism reverses orientation of punctures, changes each puncture location z to $\frac{1}{z}$, and composes local coordinate maps with I (i.e., if f is the local coordinate at the puncture z on a sphere $Q \in K$, then $f \circ I$ is the local coordinate at the puncture $1/z$ on the sphere $I(Q) \in K^*$). Finally, given $Q \in K(n)$, with local coordinates at 0 given by $(a_0^{(n)}, A^{(n)})$, renormalization via the map $(a_0^{(n)})^{-w\frac{d}{dw}}$ is required to express $I(Q)$ in its canonical form, but the isomorphism is clearest when canonical representatives are not chosen. The transformation map is functorial on equivalence classes, i.e. it commutes with actions of the symmetric groups by construction and also commutes with the sewing operation. To see that I indeed commutes with sewing, note that given S_1 and S_2, two spheres with incoming tubes, the conditions for the sewing $S_1 \;{}_i\infty_{-1}\; S_2$ to exist are precisely the same conditions required for the sewing $I(S_2) \;{}_1\infty_{-i}\; I(S_1)$ to exist and, in fact, when both exist

$$I(S_1 \;{}_i\infty_{-1}\; S_2) = I(S_2) \;{}_1\infty_{-i}\; I(S_1).$$

It is also useful to recall that $K^*(1)$ is just another name for $K(1)$, i.e. equivalence classes under conformal isomorphism of spheres of type $(1, 1)$. (But note that I is not the identity map between $K(1)$ and $K^*(1)$.)

Remark 4.4.1. *The ideal would be to eventually have a family of sets* $\mathbf{K} = \coprod_{m,n \in \mathbb{N}} K(m, n)$ *of canonical representatives of equivalence classes of (not necessarily genus-zero) worldsheets with* m *outgoing and* n *ordered incoming punctures, along with an appropriately defined sewing operation on canonical representatives. Above we have defined a sewing operation that would be appropriate for attaching* *one incoming and one outgoing puncture, but arbitrarily many incoming and outgoing punctures should be sewable under appropriate conditions. (The difficulty lies in defining the sewing operation realized on canonical representatives of equivalence classes.) In this case,* **K** *would contain both K and K^*. Such a structure would no longer be an operad but should be a type of PROP (cf. Section 1.2 of [MSS]), into which the operads K and K^* naturally embed.*

4.5 Canonical representatives in K^* under the sewing operation and the Virasoro algebra

Now that we have the added machinery of local coordinate charts and the ease of referring to a particular equivalence class of spheres, Q, as an element of $K^*(n)$, we return to the question of Section 3.3: Given $Q_1 \in K^*(m)$ and $Q_2 \in K^*(n)$ such that $Q_1 \,{}_1\infty_{-i}\, Q_2$ exist, how do we describe it as an element of $K^*(m+n-1)$, that is, how do we find a canonical representative of its equivalence class? The main obstacle to describing the local coordinate maps of $Q_1 \,{}_1\infty_{-i}\, Q_2$ in exponential notation is the fact that the operators $\{w^{j+1}\frac{d}{dw}\}_{j\in\mathbb{Z}}$ do not commute. This was overcome by a somewhat specialized approach in [H2] but since that time a much more powerful method has been introduced in [BHL]. Their result applies to any Lie algebra that has a vector space decomposition into the direct sum of two vector spaces, and in particular can be used as a kind of inverse to the Baker-Cambell-Hausdorff formula. It is applicable here because the operators $\{-w^{j+1}\frac{d}{dw}\}_{j\in\mathbb{Z}}$ give a representation of the Virasoro algebra.

More precisely, let $\mathcal{V} = \oplus_{n\in\mathbb{Z}}\mathbb{C}L_n \oplus \mathbb{C}d$ be the Virasoro algebra with the usual commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}d,$$

$$[\mathcal{V}, d] = 0$$

for $m, n \in \mathbb{Z}$. We may view $\mathcal{V}^+ = \oplus_{n \in \mathbb{Z}+} \mathbb{C} L_n$, $\mathcal{V}^- = \oplus_{n \in \mathbb{Z}+} \mathbb{C} L_{-n}$, and $\mathcal{V}_0 = \mathbb{C} L_0 \oplus \mathbb{C} d$ as Lie subalgebras of $\mathcal{V}$. Initially we consider formal variables $\{\mathcal{A}_k\}_{k \in \mathbb{Z}_+}$, $\{\mathcal{B}_k\}_{k \in \mathbb{Z}_+}$ and elements

$$g^+ = \sum_{k \in \mathbb{Z}_+} \mathcal{A}_k L_k \in \mathcal{V}^+[[\{\mathcal{A}_k\}_{k \in \mathbb{Z}_+},\ \{\mathcal{B}_k\}_{k \in \mathbb{Z}_+}]],$$

$$g^- = \sum_{k \in \mathbb{Z}_+} \mathcal{B}_k L_{-k} \in \mathcal{V}^-[[\{\mathcal{A}_k\}_{k \in \mathbb{Z}_+},\ \{\mathcal{B}_k\}_{k \in \mathbb{Z}_+}]].$$

Eventually we will be interested in convergence questions when the $\mathcal{A}_n$'s and $\mathcal{B}_n$'s are replaced by complex values, but for the time being we are interested in a type of normal ordering of the product $e^{g^-} e^{g^+}$ as expounded upon in [BHL].

By applying Corollary 4.1 and Application 4.3 in [BHL], we see that there exist unique $\Psi^+ = \sum_{k \in \mathbb{Z}_+} \Psi_k L_k$, $\Psi^- = \sum_{k \in \mathbb{Z}_+} \Psi_k L_{-k}$, Ψ_0, and Γ such that

$$e^{g^-} e^{g^+} = e^{\Psi^+} e^{\Psi_0 L_0} e^{\Psi^-} e^{\Gamma c} \tag{4.5.1}$$

where, for each $k \in \mathbb{Z}_+$,

$$\Psi_{-k} = \mathcal{B}_k + \sum_{m > k} (-k + 2m) \mathcal{A}_{-k+m} \mathcal{B}_m + P_{-k}(\mathcal{A}, \mathcal{B}),$$

$$\Psi_k = \mathcal{A}_k + \sum_{m > 0} (k + 2m) \mathcal{A}_{k+m} \mathcal{B}_m + P_k(\mathcal{A}, \mathcal{B}),$$

$$\Psi_0 = \sum_{m > 0} 2m \mathcal{A}_m \mathcal{B}_m + P_0(\mathcal{A}, \mathcal{B}),$$

$$\Gamma = \sum_{m > 0} \frac{m^3 - m}{12} \mathcal{A}_m \mathcal{B}_m + \Gamma_0(\mathcal{A}, \mathcal{B}),$$

with $P_j(\mathcal{A}, \mathcal{B}),\ \Gamma_0(\mathcal{A}, \mathcal{B}) \in \mathbb{C}[[\mathcal{A}, \mathcal{B}]]$, for $j \in \mathbb{Z}$, each containing at least one $\mathcal{A}_k$, at least one $\mathcal{B}_k$ and a total of at least three $\mathcal{A}_k$'s and $\mathcal{B}_k$'s combined.

Since these commutation relations hold for any representation of the Virasoro algebra, they are useful in our current case where L_k is represented by $-w^{k+1} \frac{d}{dw}$

in Der $\mathbb{C}$ $[w, w^{-1}]$ for $k \in \mathbb{Z}$ and d is represented by 0. Thus to solve the sewing equation, (3.3.1),

$$F^{(1)}(w) = F^{(2)}\left(g_{-j}^{-1}\left(\frac{1}{f_i(w)}\right)\right),$$

while satisfying (3.3.2), (3.3.3), and (3.3.4), we appeal to [BHL] and the following proposition which is Proposition 2.1.17 in [H2].

Proposition 4.5.1. *For any ring R, let $g(x) \in R[[x, x^{-1}]]$ and $f(x) \in R[[x]]$ with*

$$f(x) = e^{\sum_{k\in\mathbb{N}} A_k x^{k+1}\frac{d}{dx}} x.$$

Then if $g \circ f$ is well-defined,

$$g \circ f(x) = e^{\sum_{k\in\mathbb{N}} A_k x^{k+1}\frac{d}{dx}} g(x).$$

As a result if $f_i(w) = e^{\sum_{k\in Z_+} A_k w^{k+1}\frac{d}{dw}} a_0^{w\frac{d}{dw}} w$ and $g_{-j}(w) =$ $e^{-\sum_{k\in\mathbb{Z}_+} B_k x^{-k+1}\frac{d}{dx}} b_0^{-x\frac{d}{dx}} \frac{1}{x}\Big|_{x=(w^{-1}-z_{-i}^{-1})^{-1}}$, then by Proposition 4.5.1, Equation (3.3.1) may be rewritten as

$$(F^{(2)})^{-1} \circ F^{(1)}(w) = e^{\sum_{k\in Z_+} A_k w^{k+1}\frac{d}{dw}} a_0^{w\frac{d}{dw}} b_0^{w\frac{d}{dw}} e^{\sum_{k\in\mathbb{Z}_+} B_k w^{-k+1}\frac{d}{dw}} e^{z_{-i} w^2\frac{d}{dw}} w$$

Thus by (4.5.1) there exist $\{\Psi_k\}_{k\in\mathbb{Z}}$ such that

$$e^{\sum_{k\in Z_+} \Psi_{-k} w^{-k+1}\frac{d}{dw}} (a_0 b_0)^{w\frac{d}{dw}} e^{-\Psi_0 w\frac{d}{dw}} e^{\sum_{k\in Z_+} \Psi_k w^{k+1}\frac{d}{dw}} e^{z_{-i} w^2\frac{d}{dw}} w =$$
$$e^{\sum_{k\in Z_+} A_k w^{k+1}\frac{d}{dw}} a_0^{w\frac{d}{dw}} b_0^{w\frac{d}{dw}} e^{\sum_{k\in\mathbb{Z}_+} B_k w^{-k+1}\frac{d}{dw}} e^{z_{-i} w^2\frac{d}{dw}} w. \quad (4.5.2)$$

The convergence of the $\{\Psi_k\}_{k\in\mathbb{Z}}$ is shown in Corollary 4.3.2, and p. 123 of [H2] for two equivalence classes of spheres of type $(1, 1)$. Convergence then extends naturally to all of K via the inductive argument of Proposition 5.3.5 Part (5) in [H2]. Here convergence will be extended naturally from $K^*(1)$ to all of K^* in Part 5 of Proposition 8.2.2. Hence (by another application of Proposition 4.5.1) the unique solution to the sewing equation is

$$F^{(1)}(w) = e^{\sum_{k\in Z_+} \Psi_{-k} w^{-k+1}\frac{d}{dw}} w \quad (4.5.3)$$

$$F^{(2)}(w) = e^{-\sum_{k\in Z_+} \Psi_k x^{k+1}\frac{d}{dx}} (a_0 b_0 e^{\Psi_0})^{-x\frac{d}{dx}} x|_{x=(w^{-1}-z_{-i})^{-1}}. \qquad (4.5.4)$$

Remark 4.5.2. *Note that there is also a unique* Γ *corresponding to the* $\{\Psi_k\}_{k\in\mathbb{Z}}$*, which is independent of representation, but since* d *is represented by 0 here, it does not appear. Because* Γ *depends on* $\mathcal{A}$*,* $\mathcal{B}$*,* a_0*, and* b_0*, and following the notational convention of [H2], we denote* Γ *by* $\Gamma(\mathcal{A}, \mathcal{B}, a_0 b_0)$*. The definition of GVOA, as well as GVOC, may be interpreted as depending on the choice of a section of the determinant line bundle over each equivalence class of spheres with tubes, in which case* d *may be represented by the complex value that determines the section.*

Lemma 4.5.3. *Let*

$$Q_1 = (z_{-m+1}^{-1}, \ldots, z_{-1}^{-1}; A^{(-m)}, (a_0^{(-m+1)}, A^{(-m+1)}), \ldots, (a_0^{(-1)}, A^{(-1)}), (a_0^{(1)}, A^{(1)})),$$

$$Q_2 = (\zeta_{-n+1}^{-1}, \ldots, \zeta_{-1}^{-1}; B^{(-n)}, (b_0^{(-n+1)}, B^{(-n+1)}), \ldots, (b_0^{(-1)}, B^{(-1)}), (b_0^{(1)}, B^{(1)})),$$

and $1 \le i \le n$ *such that* $Q_1 \; {}_1\infty_{-i} \; Q_2$ *exists. For any* $d \in \mathbb{C}$ *the* t*-series*

$$e^{-\Gamma(\mathcal{A}^{(1)}, \mathcal{B}^{(-i)}, t^{-1} a_0^{(1)} b_0^{(-i)})d}$$

is absolutely convergent when $t = 1$*.*

The convergence result of this Lemma is shown in Lemma 5.2.1 in [H2] and is independent of the orientation of tubes not involved in the sewing. Specifically, [H2] covers the case that $b_0 = 1$ but it is clear from (4.5.2) that $a_0 b_0$ may be viewed as the first order local coordinate from Q_1 without complication. We will implicitly make use of this convergence in Step (a) of the proof of Axiom 5 of Proposition 8.2.2.

Remark 4.5.4. *In [H2], the convergence of* $\{\Psi_k\}_{k\in\mathbb{Z}}$ *and* Γ *are shown for sewings of* $K(1)$ *and then extend naturally to all sewings in* K*. But* $K(1)$ *and* $K^*(1)$ *are one and the same (i.e. conformal equivalence classes of spheres of type* $(1,1)$*), so convergence of* $\{\Psi_k\}_{k\in\mathbb{Z}}$ *and* Γ *for sewings of* $K^*(1)$ *is already shown and extends naturally to all* *of K^*. For the proof of convergence of $\{\Psi_k\}_{k\in\mathbb{Z}}$ in $K(1)$, see Corollary 4.3.2, and p. 123 in [H2].*

Using the above descriptions of $F^{(1)}$ and $F^{(2)}$ we give several examples of how to explicitly solve for these uniformaizing functions in several special cases which will be needed later:

Example 4.5.5. *Let*

$$Q_1 = (A^{(-1)}, (a_0^{(1)}, A^{(1)})) = (f_{-1}, f_1) \in K^*(1),$$

$$Q_2 = (B^{(-1)}, (b_0^{(1)}, B^{(1)})) = (g_{-1}, g_1) \in K^*(1),$$

such that $Q_1\ {}_1\infty_{-1}\ Q_2$ exists.

Appealing to [BHL] again, there exist $C^{(-1)} = \{C_j^{(-1)}\}_{j\in\mathbb{Z}}$, $C^{(1)} = \{C_j^{(1)}\}_{j\in\mathbb{Z}}$ and $c_0^{(1)}$ (which are unique and invariant under different representations of the Virasoro algebra) such that

$$e^{-\sum_{j\in Z_+} C_j^{(-1)} w^{-j+1}\frac{d}{dw}}\frac{1}{w} = e^{-\sum_{j\in Z_+}\Psi_{-j} w^{-j+1}\frac{d}{dw}} e^{-\sum_{j\in Z_+} A_j^{(-1)} w^{-j+1}\frac{d}{dw}}\frac{1}{w},$$

$$e^{\sum_{j\in Z_+} C_j^{(1)} w^{j+1}\frac{d}{dw}} (c_0^{(1)})^{w\frac{d}{dw}} w = (a_0^{(1)} e^{\Psi_0})^{w\frac{d}{dw}} e^{\sum_{j\in Z_+}\Psi_j w^{-j+1}\frac{d}{dw}} e^{\sum_{j\in Z_+} B_j^{(1)} w^{j+1}\frac{d}{dw}} (b_0^{(1)})^{w\frac{d}{dw}} w.$$

Then by the above discussion on sewing, (4.5.3) and (4.5.4) we see that

$$\begin{aligned} Q_1\ {}_1\infty_{-1}\ Q_2 &= (f_{-1}\circ (F^{(1)})^{-1}, g_1 \circ (F^{(2)})^{-1}) \\ &= (C^{(-1)}, (c_0^{(1)}, C^{(1)})). \end{aligned} \tag{4.5.5}$$

Example 4.5.6. *We now consider the sewing*

$$Q_1 = (z^{-1}; \mathbf{0}, (1, \mathbf{0}), (1, \mathbf{0})) = (z^{-1}; w^{-1}, w^{-1} - z, w) \in K^*(2),$$

$$Q_2 = (B^{(-1)}, (1, \mathbf{0})) = (g_{-1}, w) \in K^*(1),$$

such that $Q_1 \ {}_1\infty_{-1} \ Q_2$ exists. The unique $F^{(1)}$ and $F^{(2)}$ satisfying (3.3.1) are

$$\begin{aligned} F^{(1)}(w) =& g_{-1}^{-1}(w^{-1}) = e^{\sum_{j\in\mathbb{Z}_+} B_j w^{-j+1}\frac{d}{dw}} w, \\ F^{(2)}(w) =& w. \end{aligned}$$

Thus

$$Q_1 \ {}_1\infty_{-1} \ Q_2 = (g_{-1}^{-1}(z^{-1}); B^{(-1)}, (\Theta_0^{(1)}, \Theta^{(1)}), (1, \mathbf{0}))$$

where $\Theta_0^{(1)}$ and $\Theta^{(1)} = \{\Theta_j^{(1)}\}_{j\in\mathbb{Z}_+}$ are determined by

$$e^{-\sum_{j\in\mathbb{Z}_+} \Theta_j^{(1)} x^{-j+1}\frac{d}{dx}} e^{-\Theta_0^{(1)} x\frac{d}{dx}} \frac{1}{x}\bigg|_{\frac{1}{x} = \frac{1}{w} - \frac{1}{g_{-1}^{-1}(z)}} = e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-1)} w^{-j+1}\frac{d}{dw}} \frac{1}{w} - z.$$

Equivalently, let $\hat{f}_1(x) = e^{\sum_{j\in\mathbb{Z}} B_j^{(-1)} x^{j+1}\frac{d}{dx}} x$ so that

$$Q_1 \ {}_1\infty_{-2} \ Q_2 = ((\hat{f}_1(z^{-1}))^{-1}; B^{(-1)}, (\Theta_0^{(1)}, \Theta^{(1)}), (1, \mathbf{0})) \qquad (4.5.6)$$

where $\Theta_0^{(1)}$ and $\Theta^{(1)} = \{\Theta_j^{(1)}\}_{j\in\mathbb{Z}_+}$ are determined by

$$e^{-\sum_{j\in\mathbb{Z}_+} \Theta_j^{(1)} x^{-j+1}\frac{d}{dx}} e^{-\Theta_0^{(1)} x\frac{d}{dx}} \frac{1}{x}\bigg|_{\frac{1}{x} = \frac{1}{w} - \hat{f}_1^{-1}(z)} = \hat{f}_1(w^{-1}) - z. \qquad (4.5.7)$$

The above $\Theta_j^{(1)}$ are defined in terms of $B^{(-1)}$ and z and may be interpreted as a family of functions in these variables: $\Theta_j^{(1)} = \Theta_j^{(1)}(B^{(-1)}, z)$. These functions satisfy the first sewing identity (4.5.8) as seen in the following proposition.

Proposition 4.5.7. *In the algebra $(End\ \mathbb{C})[w, w^{-1}][z][[\mathcal{B}^{(-1)}]]$ (where w, z and the $\mathcal{B}_j^{(-1)}$, for $j \in \mathbb{Z}_+$, may be considered as formal variables) we have*

$$e^{\sum_{k=-1}^{\infty}\left(\sum_{j\in\mathbb{Z}_+}\binom{j+1}{k+1}\mathcal{B}_j^{(-1)} z^{j-k}\right) w^{-k+1}\frac{\partial}{\partial w}} = e^{\Theta_0^{(1)} w\frac{\partial}{\partial w}} e^{\sum_{j\in\mathbb{Z}_+} \Theta_j^{(1)} w^{-j+1}\frac{\partial}{\partial w}} e^{(z - \hat{f}_2^{-1}(z)) w^2 \frac{\partial}{\partial w}}. \qquad (4.5.8)$$

Proof. This identity may be observed from Proposition 2.2.7 of [H2]. Let $x = w^{-1}$, $y = z$, $\alpha_0 = 1$, $\mathcal{A} = \mathcal{B}^{(-1)}$ in Proposition 2.2.7 of [H2], then take the inverse of both sides of the equation. □

Example 4.5.8. *To generalize the last example, let*

$$Q_1 = (z_{-1}^{-1}; A^{(-2)}, (a_0^{(-1)}, A^{(-1)}), (1, \mathbf{0})) \in K^*(2),$$

$$Q_2 = (\zeta_{-n+1}^{-1}, \dots, \zeta_{-1}^{-1}; B^{(-n)}, (b_0^{(-n+1)}, B^{(-n+1)}), \dots, \\ (b_0^{(-1)}, B^{(-1)})), (b_0^{(1)}, B^{(1)})) \in K^*(n),$$

such that $Q_1\ {}_1\infty_{-n}\ Q_2$ *exists. Let* $\hat{f}_1$, $F^{(1)}$ *and* $F^{(2)}$ *be defined as above. Thus*

$$Q_1\ {}_1\infty_{-1}\ Q_2 = ((\hat{f}_1(z^{-1}))^{-1}, \zeta_{-n+1}^{-1}, \dots, \zeta_{-1}^{-1}; C^{(-n)}, (c_0^{(-n+1)}, C^{(-n+1)}), \\ (b_0^{(-n+1)}, B^{(-n+1)}), \dots, (b_0^{(-1)}, B^{(-1)})), (b_0^{(1)}, B^{(1)}))$$

where $c_0^{(-n+1)}$, $C^{(-n+1)} = \{C_j^{(-n+1)}\}_{j\in\mathbb{Z}_+}$, $C^{(-n)} = \{C_j^{(-n)}\}_{j\in\mathbb{Z}_+}$, *are defined by*

$$e^{-\sum_{j\in\mathbb{Z}_+} C_j^{(-n+1)} w^{-j+1}\frac{d}{dw}} (c_0^{(-n+1)})^{-w\frac{d}{dw}} \\ = e^{-\sum_{j\in\mathbb{Z}_+} \Theta_j^{(1)} w^{-j+1}\frac{d}{dw}} e^{-\Theta_0^{(1)} w\frac{d}{dw}} e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-1)} w^{-j+1}\frac{d}{dw}} (a_0^{(-1)})^{-w\frac{d}{dw}},$$

$$e^{-\sum_{j\in\mathbb{Z}_+} C_j^{(-n)} w^{-j+1}\frac{d}{dw}} = e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-n)} w^{-j+1}\frac{d}{dw}} e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-2)} w^{-j+1}\frac{d}{dw}}.$$

Note that $c_0^{(-n+1)}$, $\mathcal{C}^{(-n+1)}$ *and* $\mathcal{C}^{(-n)}$ *are uniquely determined (and independent of Virasoro representation) by [BHL].*

Example 4.5.9. *To motivate the second sewing identity (4.5.11), let*

$$Q_1 = (\mathbf{0}, (a_0^{(1)}, A^{(1)})) = (w^{-1}, f_1) \in K^*(1),$$

$$Q_2 = (\zeta^{-1}; \mathbf{0}, (1, \mathbf{0}), (1, \mathbf{0})) = (\zeta^{-1}; w^{-1}, w^{-1} - \zeta, w) \in K^*(2),$$

such that $Q_1\ {}_1\infty_{-2}\ Q_2$ *exists. The unique* $F^{(1)}$ *and* $F^{(2)}$ *satisfying (3.3.1) are*

$$F^{(1)}(w) = w$$
$$F^{(2)}(w) = f_1^{-1}(w) = (a_0^{(1)})^{-w\frac{d}{dw}} e^{-\sum_{j\in\infty} A_j^{(1)} w^{j+1}\frac{d}{dw}} w.$$

Thus

$$Q_1\ {}_1\infty_{-2}\ Q_2 = (f_1^{-1}(\zeta^{-1}); \mathbf{0}, ((a_0^{(1)})^{-1} e^{\Theta_0^{(2)}}, \Theta^{(2)}((a_0^{(1)})^{-1})), (a_0^{(1)}, A^{(1)}))$$

where $\Theta_0^{(2)}$ *and* $\Theta^{(2)} = \{\Theta_j^{(2)}\}_{j\in\mathbb{Z}_+}$ *are determined by*

$$e^{-\sum_{j\in\mathbb{Z}_+}(a_0^{(1)})^{-j}\Theta_j^{(2)}x^{-j+1}\frac{d}{dx}}(a_0^{(1)})^{x\frac{d}{dx}}e^{-\Theta_0^{(2)}x\frac{d}{dx}}\frac{1}{x}\bigg|_{\frac{1}{x}=\frac{1}{w}-\frac{1}{f_1^{-1}(\zeta)}} = e^{\sum_{j\in\mathbb{Z}_+}A_j^{(1)}w^{j+1}\frac{d}{dw}}\frac{1}{w}-\zeta.$$

Equivalently, let $\hat{f}_2(x) = e^{\sum_{j\in\mathbb{Z}}(a_0^{(1)})^{-j}A_j^{(1)}x^{-j+1}\frac{d}{dx}}x$ *so that*

$$Q_1\ {}_1\infty_{-2}\ Q_2 = ((a_0^{(1)}\hat{f}_2(\zeta))^{-1}; \mathbf{0}, ((a_0^{(1)})^{-1}e^{\Theta_0^{(2)}}, \Theta^{(2)}((a_0^{(1)})^{-1})), (a_0^{(1)}, A^{(1)})) \quad (4.5.9)$$

where $\Theta_0^{(2)}$ *and* $\Theta^{(2)} = \{\Theta_j^{(2)}\}_{j\in\mathbb{Z}_+}$ *are determined by*

$$e^{-\sum_{j\in\mathbb{Z}_+}\Theta_j^{(2)}x^{-j+1}\frac{d}{dx}}e^{-\Theta_0^{(2)}x\frac{d}{dx}}\frac{1}{x}\bigg|_{\frac{1}{x}=\frac{1}{w}-\hat{f}_2(\zeta^{-1})} = (\hat{f}_2(w))^{-1}-\zeta. \quad (4.5.10)$$

This equation may be interpretted as a rewriting of Equation (2.2.30) in [H2] from the GVOA case. The $\Theta_j^{(2)}$*, interpreted as functions in* $a_0^{(1)}$*,* $A^{(1)}$ *and* ζ *satisfy the sewing identity called the second sewing identity (4.5.11).*

Proposition 4.5.10. *In the algebra* $(End\,\mathbb{C})[w,w^{-1}][\zeta][\alpha_0^{(1)},\alpha_0^{(1)}][[\mathcal{A}^{(1)}]]$ *(where* w*,* z*,* $\alpha_0^{(1)}$ *and the* $\mathcal{A}_j^{(-1)}$*, for* $j\in\mathbb{Z}_+$*, may be considered as formal variables) we have*

$$e^{-\sum_{k=-1}^{\infty}\left(\sum_{j\in\mathbb{Z}_+}(\alpha_0^{(1)})^{-j}\mathcal{A}_j^{(1)}\binom{-j+1}{k+1}\zeta^{-j-k}\right)w^{-k+1}\frac{\partial}{\partial w}} = e^{\Theta_0^{(2)}w\frac{\partial}{\partial w}}e^{\sum_{j\in\mathbb{Z}_+}\Theta_j^{(2)}w^{-j+1}\frac{\partial}{\partial w}}e^{(\zeta-\hat{f}_2(\zeta))w^2\frac{\partial}{\partial w}}. \quad (4.5.11)$$

Proof. This identity is observed from Proposition 2.2.9 of [H2] as follows. Let $x = w^{-1}$, $y = \zeta$, and $\mathcal{B} = \mathcal{A}^{(1)}((a_0^{(1)})^{-1})$ in Proposition 2.2.9, then take the inverse of both sides of the equation. □

Example 4.5.11. *Again we generalize and let*

$$Q_1 = (A^{(-1)}, (a_0^{(1)}, A^{(1)})) \in K^*(1),$$

$$Q_2 = (\zeta^{-1}; \mathbf{0}, (b_0^{(-1)}, B^{(-1)}), (b_0^{(1)}, B^{(1)})) \in K^*(2),$$

such that $Q_1\ {}_1\infty_{-2}\ Q_2$ exists. The unique $F^{(1)}$ and $F^{(2)}$ satisfying (3.3.1) are as in the previous example. Similarly, define $\hat{f}_2$ as above. Thus

$$Q_1\ {}_1\infty_{-2}\ Q_2 = ((a_0^{(1)}\hat{f}_2(\zeta))^{-1}; A^{(-1)}, (c_0^{(-1)}, C^{(-1)}), (c_0^{(1)}, \mathcal{C}^{(1)})) \tag{4.5.12}$$

where $c_0^{(i)}$, $C^{(i)} = \{C_j^{(i)}\}_{j\in\mathbb{Z}+}$, for $i = -1, 1$ are defined by

$$e^{-\sum_{j\in\mathbb{Z}_+} C_j^{(-1)} w^{-j+1}\frac{d}{dw}} (c_0^{(-1)})^{-w\frac{d}{dw}}$$
$$= e^{-\sum_{j\in\mathbb{Z}_+}(a_0^{(1)})^{-j}\Theta_j^{(2)} w^{-j+1}\frac{d}{dw}} \left(\frac{e^{\Theta_0^{(2)}}}{a_0^{(1)}}\right)^{-w\frac{d}{dw}} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-1)} w^{-j+1}\frac{d}{dw}} (b_0^{(-1)})^{-w\frac{d}{dw}},$$

$$e^{\sum_{j\in\mathbb{Z}_+} C_j^{(1)} w^{j+1}\frac{d}{dw}} (c_0^{(1)})^{w\frac{d}{dw}} = e^{\sum_{j\in\mathbb{Z}_+} A_j^{(1)} w^{j+1}\frac{d}{dw}} (a_0^{(1)})^{w\frac{d}{dw}} e^{\sum_{j\in\mathbb{Z}_+} B_j^{(1)} w^{j+1}\frac{d}{dw}} (b_0^{(1)})^{w\frac{d}{dw}}.$$

Note that $c_0^{(i)}$ and $C^{(i)}$ are uniquely determined, for $i = -1, 1$ (and independent of Virasoro representation).

This concludes the examples of sewing we will do explicitly.

4.6 The linear functionals $\mathcal{L}_I(z)$

In our work to understand representations of the Virasoro algebra in a given vertex operator coalgebra, a particular family of linear functionals will be of interest. Similar linear functionals are used in [H2] (Proposition 3.2.5 and the preceding discussion) for the VOA case. First we define the notion of a meromorphic function on any manifold $K^*(n)$, for $n \in \mathbb{N}$.

Definition 4.6.1. *A meromorphic function on $K^*(n)$ is a function that, when viewed as a function of z_{-i}, for $i = 1, \ldots, n-1$, of $a_0^{(-i)}$, for $i = -1, 1, \ldots, n-1$, and of $A_j^{(-i)}$, for $i = -1, 1, \ldots, n$ and $j \in \mathbb{Z}_+$, is a polynomial in these variables divided by the product of powers of $a_0^{(-i)}$, for $i = -1, 1, \ldots, n-1$, of z_{-i}, for $i = 1, \ldots, n-1$,* *and of* $z_{-i} - z_{-j}$*, for* $1 \leq i < j \leq n-1$ *(cf. [H2] p. 67). Explicitly, if* F *is a meromorphic function on* $K^*(n)$ *and* $Q \in K^*(n)$ *is as in (3.5.4) or (3.5.6), then*

$$F(Q) \quad = \quad \frac{\tilde{F}(z_{-n+1},\ldots,z_{-1},\mathcal{A}^{(-n)},\ldots,\mathcal{A}^{(-1)},\mathcal{A}^{(1)},a_0^{(-n+1)},\ldots,a_0^{(-1)},a_0^{(1)})}{(a_0^{(1)})^{r_1}\prod_{i=1}^{n-1}(a_0^{(-i)})^{r_{-i}}\prod_{i<j}(z_{-i}-z_{-j})^{s_{ij}}\prod_{i=1}^{n-1}z_{-i}^{t_{-i}}},$$

where $\tilde{F}$ *is a polynomial, and* $r_i, s_{ij}, t_i \in \mathbb{N}$.

The linear functionals of interest to us will be denoted $\mathcal{L}_I(z)$, $z \in \mathbb{C}^\times$, and will map from the space of meromorphic functions on $K^*(1)$ to $\mathbb{C}$. We define these linear functionals by

$$\mathcal{L}_I(z)F = \left(\frac{d}{d\varepsilon}F\left(((1,(0,-\varepsilon,0,0,\ldots)))_1\infty_{-1}(z^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0}))\right)\right)\Bigg|_{\varepsilon=0}.$$

Since $\mathcal{L}_I(z)$ may be thought of as an element of the tangent space of $K^*(1)$ at $(\mathbf{0},(1,\mathbf{0}))$, it may be rewritten

$$\mathcal{L}_I(z) = \sum_{k\in\mathbb{Z}_+}\gamma_{-k}\frac{\partial}{\partial A_k^{(-1)}}\Bigg|_{A^{(-1)}=\mathbf{0}} + \gamma_0\frac{\partial}{\partial a_0^{(1)}}\Bigg|_{a_0^{(1)}=1} + \sum_{k\in\mathbb{Z}_+}\gamma_k\frac{\partial}{\partial A_k^{(1)}}\Bigg|_{A^{(1)}=\mathbf{0}}$$

for some coefficients γ_k, with $k \in \mathbb{Z}$. In fact, we can completely describe these coefficients as the following lemma shows.

Lemma 4.6.2.

$$\mathcal{L}_I(z) = \sum_{k\in\mathbb{Z}_+} -z^{-k-2}\ \frac{\partial}{\partial A_k^{(-1)}}\Bigg|_{A^{(-1)}=\mathbf{0}} -\ z^{-2}\frac{\partial}{\partial a_0^{(1)}}\Bigg|_{a_0^{(1)}=1} + \sum_{k\in\mathbb{Z}_+} -z^{k-2}\frac{\partial}{\partial A_k^{(1)}}\Bigg|_{A^{(1)}=\mathbf{0}}.$$

Proof. The idea of the proof will be to sew $((1,(0,-\varepsilon,0,0,...)))$ and $(z^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0}))$ together then examine what the derivative with respect to ε at $\varepsilon = 0$ is for each of the local coordinate coefficients. In some ways this reflects the approach in [H2] to a similar linear functional $\mathcal{L}_I(z)$, but here we are forced to consider nontrivial scaling at the outgoing puncture and also the technique we use avoids the unresolved branching issue in (3.2.10) and (3.2.11) of [H2].

First, we denote the local coordinate map of $((1,(0,-\varepsilon,0,0,...)))$ as $f_1(w)$ and the local coordinate maps of $(z^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0}))$ as $g_i(w)$ where the subscript indicates which puncture the local coordinate map pertains to. Thus we get the sewing maps $F^{(1)}(w)=w$ and $F^{(2)}(w)=f_1^{-1}\left(\frac{1}{g_{-1}(w)}\right)=e^{zw^2\frac{d}{dw}}e^{\varepsilon w^3\frac{d}{dw}}w$. These maps move the last outgoing puncture to $k=e^{\varepsilon x^3\frac{d}{dx}}x\Big|_{x=-z^{-1}}$ so we employ the transformation map $T_1(w)=e^{k^{-1}w^2\frac{d}{dw}}w$. (Notice this does not guarantee that we will have a canonical representative of an element in $K^*(1)$ since there may be a nontrivial linear scaling of the local coordinate map at ∞.) After sewing and applying T_1, the local coordinate map at 0 is $g_1\circ(F^{(2)})^{-1}\circ T_1^{-1}(w)$ and the local coordinate map at ∞ is $g_{-2}\circ(F^{(2)})^{-1}\circ T_1^{-1}(w)$, both of which depend on ε and will be trivial if $\varepsilon=0$.

Now, we take the derivative of the puncture at 0 with respect to ε and evaluate at $\varepsilon=0$ (using Proposition 4.5.1 twice).

$$\frac{d}{d\varepsilon}\left(g_1\circ(F^{(2)})^{-1}\circ T_1^{-1}(w)\right)\Big|_{\varepsilon=0}=\frac{d}{d\varepsilon}\left(e^{-k^{-1}w^2\frac{d}{dw}}e^{-\varepsilon w^3\frac{d}{dw}}e^{-zw^2\frac{d}{dw}}w\right)\Big|_{\varepsilon=0}$$

$$=\frac{d}{d\varepsilon}\left(e^{-(-z^{-1}-\varepsilon z^{-3}-\ldots)^{-1}w^2\frac{d}{dw}}((w^{-1}+z)^{-1}-\varepsilon w(w^{-1}+z)^{-2}+\ldots)\right)\Big|_{\varepsilon=0}$$

$$=\frac{d}{d\varepsilon}\Big(((w^{-1}-(z^{-1}+\varepsilon z^{-3}+\ldots)^{-1})+z)^{-1}-\varepsilon(w^{-1}-(z^{-1}+\varepsilon z^{-3}+\ldots)^{-1})^{-1}((w^{-1}-(z^{-1}+\varepsilon z^{-3}+\ldots)^{-1})+z)^{-2}+\ldots\Big)\Big|_{\varepsilon=0} \quad (4.6.1)$$

We only use "..." for terms involving higher powers of ε, which will disappear upon differentiating and evaluating at $\varepsilon=0$, so this notation is unambiguous. Now applying the derivative and evaluating, the right-hand side of (4.6.1) at $\varepsilon=0$, we obtain

$$-(w^{-1}-z+z)^{-2}z^2z^{-3}-(w^{-1}-z)^{-1}(w^{-1}-z+z)^{-2}$$

$$=-w^2z^{-1}-w^2(w^{-1}-z)^{-1}$$
$$=-w^2\frac{z^{-1}}{1-wz}$$

We are dealing with local coordinates at 0 which means there will be no negative powers of w. Thus

$$\frac{d}{d\varepsilon}\left(g_1 \circ (F^{(2)})^{-1} \circ T_1^{-1}(w)\right)\Big|_{\varepsilon=0} = -w^2 \frac{z}{1-wz} = -\sum_{k=0}^{\infty} z^{k-1} w^{k+2}. \tag{4.6.2}$$

Using the same approach for the local coordinates at ∞, we find that

$$\begin{aligned}\frac{d}{d\varepsilon}\left(g_{-2} \circ (F^{(2)})^{-1} \circ T_1^{-1}(w)\right)\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\left(\frac{1}{(F^{(2)})^{-1} \circ T_1^{-1}(w)}\right)\Bigg|_{\varepsilon=0} \\ &= -((F^{(2)})^{-1} \circ T_1^{-1}(w))^{-2} \frac{d}{d\varepsilon}\left((F^{(2)})^{-1} \circ T_1^{-1}(w)\right)\Bigg|_{\varepsilon=0} \\ &= -w^{-2}\left(-w^2 \frac{z^{-1}}{1-wz}\right) \\ &= -\frac{w^{-1}z^{-2}}{1-w^{-1}z^{-1}} \\ &= -\sum_{k=0}^{\infty} z^{-k-2} w^{-k-1}. \end{aligned} \tag{4.6.3}$$

We see now that the local coordinates at ∞ do have an a_0 term that depends on ε since their derivative evaluated at $\varepsilon = 0$ has a nontrivial (not equal to 1) w^{-1} term, and we also see that since we have a trivial w term, the local coordinates at 0 have a trivial a_0 term. We want to find out what the coefficients are in front of the canonical tangent vectors, though, so let $h_1(w) = e^{\sum_{k \in \mathbb{Z}_+} A_k^{(1)}(\varepsilon) w^{k+1} \frac{d}{dw}} (a_0^{(1)}(\varepsilon))^{w \frac{d}{dw}} w$ and $h_{-1}(w) = e^{\sum_{k \in \mathbb{Z}_+} -A_k^{(-1)}(\varepsilon) w^{-k+1} \frac{d}{dw}} \frac{1}{w}$ be the canonical representatives of the local coordinates obtained from the sewing (where $A_k^{(i)}(\varepsilon), a_0^{(1)}(\varepsilon)$ are functions depending on ε for $i = -1, 1$ and $k \in \mathbb{Z}_+$). By applying the transformation map $T_2(w) = (a_0^{(1)}(\varepsilon))^{w \frac{d}{dw}} w$, we again obtain the unique local coordinates with no nontrivial a_0 at 0 and a possibly nontrivial a_0 term at ∞. Taking the derivative of these local coordinates with respect to ε at $\varepsilon = 0$ yields

$$\begin{aligned}
&\frac{d}{d\varepsilon}\left(h_1 \circ T_2^{-1}(w)\right)\Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\left((a_0^{(1)}(\varepsilon))^{-w\frac{d}{dw}} e^{\sum_{k\in\mathbb{Z}_+} A_k^{(1)}(\varepsilon)w^{k+1}\frac{d}{dw}} (a_0^{(1)}(\varepsilon))^{w\frac{d}{dw}} w\right)\Bigg|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\left(e^{\sum_{k\in\mathbb{Z}_+} (a_0^{(1)}(\varepsilon))^{-k} A_k^{(1)}(\varepsilon)w^{k+1}\frac{d}{dw}} (a_0^{(1)}(\varepsilon))^{-w\frac{d}{dw}} (a_0^{(1)}(\varepsilon))^{w\frac{d}{dw}} w\right)\Bigg|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\left(e^{\sum_{k\in\mathbb{Z}_+} (a_0^{(1)}(\varepsilon))^{-k} A_k^{(1)}(\varepsilon)w^{k+1}\frac{d}{dw}} w\right)\Bigg|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\left(w + \sum_{k\in\mathbb{Z}_+} (a_0^{(1)}(\varepsilon))^{-k} A_k^{(1)}(\varepsilon)w^{k+1} + \dots\right)\Bigg|_{\varepsilon=0} \\
&= \sum_{k\in\mathbb{Z}_+}\left(\frac{d}{d\varepsilon}(a_0^{(1)}(\varepsilon))^{-k} A_k^{(1)}(\varepsilon)\right)\Bigg|_{\varepsilon=0} w^{k+1} \\
&= \sum_{k\in\mathbb{Z}_+}\left(\frac{d}{d\varepsilon} A_k^{(1)}(\varepsilon)\right)\Bigg|_{\varepsilon=0} w^{k+1}
\end{aligned} \tag{4.6.4}$$

for the puncture at 0 and

$$\begin{aligned}
&\frac{d}{d\varepsilon}\left(h_{-1} \circ T_2^{-1}(w)\right)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\left((a_0^{(1)}(\varepsilon))^{-w\frac{d}{dw}} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-1)}(\varepsilon)w^{-k+1}\frac{d}{dw}} \frac{1}{w}\right)\Bigg|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\left(e^{-\sum_{k\in\mathbb{Z}_+} (a_0^{(1)}(\varepsilon))^{k} A_k^{(-1)}(\varepsilon)w^{-k+1}\frac{d}{dw}} (a_0^{(1)}(\varepsilon))^{-w\frac{d}{dw}} \frac{1}{w}\right)\Bigg|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\left((a_0^{(1)}(\varepsilon))w^{-1} + \sum_{k\in\mathbb{Z}_+} (a_0^{(1)}(\varepsilon))^{k+1} A_k^{(-1)}(\varepsilon)w^{-k-1} + \dots\right)\Bigg|_{\varepsilon=0} \\
&= \left(\frac{d}{d\varepsilon}(a_0^{(1)}(\varepsilon))\right)\Bigg|_{\varepsilon=0} w^{-1} + \sum_{k\in\mathbb{Z}_+}\left(\frac{d}{d\varepsilon} A_k^{(-1)}(\varepsilon)\right)\Bigg|_{\varepsilon=0} w^{-k-1}
\end{aligned} \tag{4.6.5}$$

for the puncture at ∞.

By comparing the coefficients of (4.6.2) and (4.6.4) as well as (4.6.3) and (4.6.5), we verify the claim of the lemma. □

CHAPTER 5

ALGEBRAIC PRELIMINARIES

In the examples of $\mathcal{C}irc$-algebras and -coalgebras, it was insightful to study the formal algebraic structure that a geometric structure induces on vector spaces. In order to study the kind of structure that K^* induces on vector spaces, it is necessary to define and investigate some algebraic formalism. This section will cover the formal calculus we will need. For a more thorough exposition on formal calculus itself, see [FLM], [FHL] or [LL].

5.1 The δ-function

We will use the "formal δ-function", $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$, which is discussed in, for instance, [FHL]. Note that the δ-function applied to $\frac{x_1-x_2}{x_0}$, where x_0, x_1 and x_2 are commuting formal variables, is a formal power series in x_2 (i.e., negative powers of $(x_1 - x_2)$ are expanded in nonnegative integral powers of x_2). In general, in the formal calculus of VOAs the sum or difference of two formal variables, $(x_1 \pm x_2)$, is understood to be expanded in nonnegative integral powers of the second, x_2 (cf. [FLM], [FHL]).

The following three properties of δ-functions will be relevant:

First, given a formal Laurent series $X(x_1, x_2) \in (\mathrm{Hom}(V, W))[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$ with coefficients which are homomorphisms from a vector space V to a vector space W, if $\lim_{x_1 \to x_2} X(x_1, x_2)$ exists (i.e. when $X(x_1, x_2)$ is applied to any element of V, setting $x_1 = x_2$ leads to only finite sums in W) we have

$$\delta\left(\frac{x_1}{x_2}\right) X(x_1, x_2) = \delta\left(\frac{x_1}{x_2}\right) X(x_2, x_2). \tag{5.1.1}$$

Second, we use the fact that

$$x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) \tag{5.1.2}$$

which is proved by direct expansion or a more conceptual calculation in [LL]. The third fact,

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right). \tag{5.1.3}$$

may be observed similarly. These facts are used freely throughout VOA theory.

5.2 Linear algebra on $\mathbb{Z}$-graded vector spaces with finite-dimensional homogeneous subspaces

Let $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$ be a $\mathbb{Z}$-graded vector space over $\mathbb{C}$ such that $\dim V_{(n)} < \infty$ for each $n \in \mathbb{Z}$. We denote the graded dual space of V by

$$V' = \coprod_{n \in \mathbb{Z}} V_{(n)}^*,$$

the algebraic closure of V by

$$\overline{V} = \prod_{n \in \mathbb{Z}} V_{(n)} = (V')^*,$$

and the natural pairing of V' with $\overline{V}$ by $\langle \cdot, \cdot \rangle$. The n-th tensor product of V, denoted $V^{\otimes n}$, is still a $\mathbb{Z}$-graded vector space (where $v \in V_{(k_1)} \otimes \ldots \otimes V_{(k_n)}$ has weight $k_1 + \ldots + k_n$) with finite-dimensional homogeneous subspaces. Thus $(V^{\otimes n})'$, $\overline{V^{\otimes n}}$ and $\langle \cdot, \cdot \rangle : (V^{\otimes n})' \times \overline{V^{\otimes n}} \to \mathbb{C}$ are defined as above.

We denote a homogeneous basis of V by

$$\{e_{\ell^{(k)}}^{(k)} | k \in \mathbb{Z},\ \ell^{(k)} = 1, \ldots, \dim V_{(k)}\}$$

and its corresponding dual basis of V' by

$$\{(e^{(k)}_{\ell^{(k)}})^* | k \in \mathbb{Z},\ \ell^{(k)} = 1, \ldots, \dim V_{(k)}\}.$$

We will use the notation

$$\mathcal{H}_V(m,n) = \mathrm{Hom}(V^{\otimes m}, \overline{V^{\otimes n}})$$

for $m, n \in \mathbb{N}$. For $m \in \mathbb{N}$, $n \in \mathbb{Z}_+$ and any integer $0 < i \leq n$, we define the *t-contraction* by

$$\begin{aligned}(\cdot\ {}_1*_{-i}\ \cdot)_t : \mathcal{H}_V(1,m) \times \mathcal{H}_V(1,n) &\to \mathrm{Hom}(V, \overline{V^{\otimes m+n-1}}[[t,t^{-1}]]) \\ (f,g) &\mapsto (f\ {}_1*_{-i}\ g)_t\end{aligned}$$

where

$$(f\ {}_1*_{-i}\ g)_t v = \sum_{k\in\mathbb{Z}} \sum_{\ell^{(k)}=1}^{\dim V_{(k)}} \Bigg(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{i-1} \otimes f(e^{(k)}_{\ell^{(k)}}) \cdot (e^{(k)}_{\ell^{(k)}})^* \otimes \\ \underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-i} \Bigg) g(v) t^k.$$

If for all $v' \in (V^{\otimes n+m-1})'$, $v \in V$ the formal Laurent series

$$\langle v', (f\ {}_1*_{-i}\ g)_t v\rangle$$

is absolutely convergent when $t = 1$, then $(f\ {}_1*_{-i}\ g)_1$ is well-defined as an element of $\mathcal{H}_V(1, m+n-1)$ and we define the *contraction* of f and g by

$$f\ {}_1*_{-i}\ g = (f\ {}_1*_{-i}\ g)_1 \in \mathcal{H}_V(1, m+n-1).$$

The following associativity of t-contractions comes from the definition.

Proposition 5.2.1. *Let $\ell, m \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ such that $m+n > 1$. Choose $f_1 \in \mathcal{H}_V(1,\ell)$, $f_2 \in \mathcal{H}_V(1,m)$, $f_3 \in \mathcal{H}_V(1,n)$ and integers $1 \leq i \leq m+n-1$ and $1 \leq j \leq n$. One of the following three properties holds:*

(1) $i < j$ *and as a formal series in* t_1 *and* t_2

$$(f_1 \;{}_1*_{-i}\; (f_2 \;{}_1*_{-j}\; f_3)_{t_1})_{t_2} = (f_2 \;{}_1*_{-j-\ell+1}\; (f_1 \;{}_1*_{-i}\; f_3)_{t_2})_{t_1};$$

(2) $j \leq i < j+m$ *and as a formal series in* t_1 *and* t_2

$$(f_1 \;{}_1*_{-i}\; (f_2 \;{}_1*_{-j}\; f_3)_{t_1})_{t_2} = ((f_1 \;{}_1*_{-i+j-1}\; f_2)_{t_2} \;{}_1*_{-j}\; f_3)_{t_1};$$

(3) $i \geq j+m$ *and as a formal series in* t_1 *and* t_2

$$(f_1 \;{}_1*_{-i}\; (f_2 \;{}_1*_{-j}\; f_3)_{t_1})_{t_2} = (f_2 \;{}_1*_{-j}\; (f_1 \;{}_1*_{-i+m-1}\; f_3)_{t_2})_{t_1}.$$

Since this proposition shows equality as formal power series, it also implies absolute convergence of both sides and equality given the absolute convergence of either side.

Moving from associativity to permutations, the symmetric group on n letters acts naturally on $\overline{V^{\otimes n}}$ from the left, i.e. it is given by

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

for all $\sigma \in S_n$ and $v_1, \ldots, v_n \in V$. This induces a left action on $\mathcal{H}_V(1,n)$ given by

$$\sigma(f)(v) = \sigma(f(v)).$$

Transpositions play a fundamental role in the actions of the symmetric group so we will make heavy use of the transposition map

$$T : V \otimes V \to V \otimes V$$
$$v \otimes w \mapsto w \otimes v.$$

From the definition of t-contraction and the action of the symmetric group we see the following.

Proposition 5.2.2. *Let* $f_1, f_2 \in \mathcal{H}_V(1,2)$. *Then* $f_1 \;{}_1\infty_{-1}\; f_2$ *exists if and only if* $(f_1 \;{}_1*_{-2}\; (T\; f_2))$ *also exists. If this is the case,*

$$f_1 \;{}_1*_{-1}\; f_2 = (Id_V \otimes T)(T \otimes Id_V)(f_1 \;{}_1*_{-2}\; (T\; f_2)).$$

Proposition 5.2.2 implies that all symmetric groups act functorially with respect to contraction since every symmetric group is generated by transpositions. In addition, as was the case with the moduli space of spheres with 1 incoming and n outgoing punctures, Propositions 5.2.1 and 5.2.2 imply that the sets $\mathcal{H}_V(1,n)$, $n \in \mathbb{N}$, form a partial pseudo-operad.

Remark 5.2.3. *In his book [H2], Huang examines a similar t-contraction and contraction on the set of linear maps $\mathcal{H}_V(m,1)$ for $m \in \mathbb{Z}$. The vector spaces $\mathcal{H}_{V'}(m,1)$ and $\mathcal{H}_V(1,m)$ are naturally isomorphic, and it can even be shown that the t-contraction and contraction in $\mathcal{H}_{V'}(m,1)$ correspond to the t-contraction and contraction in $\mathcal{H}_V(1,m)$ under this isomorphism. Geometric vertex operator algebras arise from considering maps $\nu_m : K(m) \to \mathcal{H}_V(m,1)$ such that the diagram*

$$\begin{array}{ccc} K(m) \times K(n) & \xrightarrow{\nu_m \times \nu_n} & \mathcal{H}_V(m,1) \times \mathcal{H}_V(n,1) \\ {}_i\infty_{-1} \big\downarrow & & {}_i*_{-1} \big\downarrow \\ K(m+n-1) & \xrightarrow{\nu_{m+n-1}} & \mathcal{H}_V(m+n-1,1) \end{array}$$

commutes (up to scalars, when the map ${}_i\infty_{-1}$ is defined). Among other conditions for GVOAs, the grading and meromorphicity axioms place additional requirements on the image of $\langle v', \nu_m(Q)v \rangle$ that depend on the choice of elements $Q \in K(m)$ and $v \in V^{\otimes m}$, but not on $v' \in V'$. In some sense we could use the isomorphisms $K(m) \cong K^*(m)$ (recall Section 4.4) and $\mathcal{H}_{V'}(m,1) \cong \mathcal{H}_V(1,m)$ to achieve the diagram

$$\begin{array}{ccc} K^*(m) \times K^*(n) & \xrightarrow{\mu_m \times \nu_n} & \mathcal{H}_V(1,m) \times \mathcal{H}_V(1,n) \\ {}_1\infty_{-i} \big\downarrow & & {}_1*_{-i} \big\downarrow \\ K^*(m+n-1) & \xrightarrow{\mu_{m+n-1}} & \mathcal{H}_V(1,m+n-1). \end{array}$$

This diagram underlies geometric vertex operator coalgebras. However, in order to maintain consistency with the current worldsheet model, GVOCs must still place additional requirements on the image of $\langle v', \mu_m(Q)v \rangle$ that depend on the choice of elements $Q \in K^*(m)$ and $v \in V$, but not on $v' \in (V^{\otimes m})'$. These requirements *would be dualized under the above isomorphisms and hence would be incorrect. The approach we provide in Chapter 8 yields a constructive isomorphism with a constructive inverse (that is natural with respect to the formal definition of K^*). The above approach is nonconstructive so that, even if the discrepancies with the grading and meromorphicity axioms were resolved, our approach produces a more useful outcome.*

5.3 More on sewing identities in representations of the Virasoro algebra

Now that we have developed another object on which the Virasoro algebra may act, it is important to highlight the independence of the choice of the Θ sequences (c.f. (4.5.7) and (4.5.10)) from the representation in which they are selected. Without specifying the specific module action of the Virasoro algebra on a complex vector space V, we can claim the following two propositions. They correspond to Propositions 4.3.9 and 4.3.10, respectively, from [H2] with the only substantial change being variable names.

Proposition 5.3.1. *Let $\Theta^{(1)} = \Theta^{(1)}(\mathcal{B}(t), y)$, $\Theta_0^{(1)} = \Theta_0^{(1)}(\mathcal{B}(t), y)$ be chosen as in Proposition 4.5.7 to depend on $\mathcal{B}(t)$, and y (with t an additional formal variable). Then in the algebra $(End\ V)[[t]][y, y^{-1}][[\mathcal{B}^{(-1)}]]$ we have*

$$e^{-\sum_{k=-1}^{\infty}\left(\sum_{j\in\mathbb{Z}_+}\binom{j+1}{k+1}\mathcal{B}_j^{(-1)}t^jy^{j-k}\right)L(-k)} = e^{-\Theta_0^{(1)}L(0)}e^{-\sum_{j\in\mathbb{Z}_+}\Theta_j^{(1)}L(-j)}e^{(-y+\hat{f}_1^{-1}(y))L(1)}.$$

Proposition 5.3.2. *Let $\Theta^{(2)} = \Theta^{(2)}(\mathcal{A}, \alpha_0^{(1)}t^{-1}, y)$, $\Theta_0^{(2)} = \Theta_0^{(2)}(\mathcal{A}, \alpha_0^{(1)}t^{-1}, y)$ be chosen as in Proposition 4.5.10 to depend on $\mathcal{A}$, $\alpha_0^{(1)}$, t and y.*

Then in the algebra $(End\ V)[[t]][y, y^{-1}][\alpha_0^{(1)}, \alpha_0^{(1)}][[\mathcal{A}^{(1)}]]$ we have

$$e^{\sum_{k=-1}^{\infty}\left(\sum_{j\in\mathbb{Z}_+}(\alpha_0^{(1)})^{-j}\mathcal{A}_j^{(1)}\binom{-j+1}{k+1}y^{-j-k}\right)L(-k)} = e^{-\Theta_0^{(2)}L(0)}e^{-\sum_{j\in\mathbb{Z}_+}\Theta_j^{(2)}L(-j)}e^{(-y+\hat{f}_2(y))L(1)}.$$

CHAPTER 6

GEOMETRIC VERTEX OPERATOR COALGEBRAS

6.1 The definition of a geometric vertex operator coalgebra

We are finally ready to define the primary geometric motivation for vertex operator coalgebras. The definition of a geometric vertex operator coalgebra closely follows the definition of a geometric vertex operator algebra first given in [H1] and [H2] (in order that they might eventually be combined). One may think of a geometric vertex operator algebra as a meromorphic morphism (or algebra) of a $\mathbb{C}$-extension of the partial operad K to the partial pseudo-operad $\{\mathcal{H}_V(n,1)\}_{n\in\mathbb{N}}$. (See Section 6 of [HL1].) Similarly, a geometric vertex operator coalgebra may be interpreted as a meromorphic morphism (or coalgebra) of a $\mathbb{C}$-extension of the partial operad K^* to the partial pseudo-operad $\{\mathcal{H}_V(1,n)\}_{n\in\mathbb{N}}$. We will not focus on this interpretation here.

Definition 6.1.1. *A* geometric vertex operator coalgebra *is a $\mathbb{Z}$-graded vector space over $\mathbb{C}$*

$$V = \coprod_{k\in\mathbb{Z}} V_{(k)}$$

such that $\dim V_{(k)} < \infty$ *for $k \in \mathbb{Z}$, together with a linear map for each $n \in \mathbb{N}$*

$$\mu_n : K^*(n) \mapsto \mathcal{H}_V(1,n)$$

satisfying the following axioms:

1. Positive energy:

$$V_{(k)} = 0 \text{ for } k \text{ sufficiently small.}$$

2. Grading: Let $k \in \mathbb{Z}$, $v \in V_{(k)}$, $v' \in V'$, and $a \in \mathbb{C}^\times$. Then

$$\langle v', \mu_1(\mathbf{0}, (a, \mathbf{0}))v\rangle = a^{-k}\langle v', v\rangle.$$

3. Meromorphicity: For any $n \in \mathbb{N}$, $v' \in (V^{\otimes n})'$, and $v \in V$ the function

$$Q \mapsto \langle v', \mu_n(Q)v\rangle$$

from $K^*(n)$ to $\mathbb{C}$ is a meromorphic function on $K^*(n)$. Further, for any $v \in V$ there exists $N(v) \in \mathbb{Z}_+$ such that for all $v' \in (V^{\otimes n})'$ the degree of z_{-i} in $\langle v', \mu_n(Q)v\rangle$ is less than $N(v)$, for $i = 1, \ldots, n-1$.

4. Permutation: Let $\sigma \in S_n$. Then for any $Q \in K^*(n)$

$$\sigma(\mu_n(Q)) = \mu_n(\sigma(Q)).$$

5. Sewing: There exists a unique complex number C (the central charge *or* rank*) such that for*

$$Q_1 = (z^{-1}_{-m+1}, \ldots, z^{-1}_{-1}; A^{(-m)}, (a_0^{(-m+1)}, A^{(-m+1)}), \ldots, (a_0^{(-1)}, A^{(-1)}),\\ (a_0^{(1)}, A^{(1)})) \in K^*(m),$$

$$Q_2 = (\zeta^{-1}_{-n+1}, \ldots, \zeta^{-1}_{-1}; B^{(-n)}, (b_0^{(-n+1)}, B^{(-n+1)}), \ldots, (b_0^{(-1)}, B^{(-1)}),\\ (b_0^{(1)}, B^{(1)})) \in K^*(n),$$

if the i-th outgoing puncture $(1 \leq i \leq n)$ of Q_2 can be sewn to the incoming puncture of Q_1, then for any $v' \in (V^{\otimes m+n-1})', v \in V$

$$\langle v', (\mu_m(Q_1) \ {}_1{*}_{-i} \ \mu_n(Q_2)_t(v)\rangle$$

is absolutely convergent when $t = 1$, and

$$\mu_{m+n-1}(Q_1 \ {}_1\infty_{-i} \ Q_2) = (\mu_n(Q_1)_1 *_{-i} \mu_m(Q_2) e^{-\Gamma(A^{(1)}, B^{(-i)}, a_0^{(1)} b_0^{(-i)})C} \tag{6.1.1}$$

where C is the rank and $\Gamma(A^{(1)}, B^{(-i)}, a_0^{(1)} b_0^{(-i)})$ is as in (4.5.1) and converges by Lemma 4.5.3.

We denote the geometric vertex operator coalgebra just defined by

$$(V, \mu = \{\mu_n\}_{n \in \mathbb{N}})$$

or, when there is no ambiguity, simply by V. We also sometimes refer to a geometric vertex operator coalgebra with central charge C, or rank C, when the central charge is of interest.

Remark 6.1.2. *The second half of the meromorphicity axiom may be interpreted as saying that $\langle v', \mu_n(Q)v \rangle$, when viewed as a function in $z_{-1}^{-1}, \ldots, z_{-n+1}^{-1}$, has poles at $z_{-i}^{-1} = 0$ (for $i = 1, \ldots, n-1$) and $z_{-i}^{-1} = z_{-j}^{-1}$ (for $i \neq j$) such that the order of each of these poles is bounded by $N(v)$ independent of $v' \in (V^{\otimes n})'$. This fact comes directly from the meromorphicity axiom for poles at $z_i^{-1} = 0$ but for poles at $z_i^{-1} = z_j^{-1}$ this observation involves the sewing axiom. Incidently, the meromorphicity axiom of a geometric vertex operator algebra includes a small bit of redundancy in that the order of the poles at $z_i = z_j$ (for $i \neq j$) is bounded as a result of the bounding of poles at $z_i = 0$ (for $i = 1, \ldots, n-1$) and the sewing axiom.*

CHAPTER 7

VERTEX OPERATOR COALGEBRAS

The following description of a vertex operator coalgebra is the central definition of this thesis. It is in a sense the culmination of our effort to algebraically understand the structure induced on vector spaces by moduli spaces of spheres with outgoing tubes. In another sense, however, it is a starting point for algebraic study. A number of algebraic properties will be described in this section but these barely scratch the surface in reference to questions about the structure of vertex operator coalgebras and representations over them. Here is the definition.

7.1 The definition of a vertex operator coalgebra

Definition 7.1.1. *A* vertex operator coalgebra (over $\mathbb{C}$) (of central charge or rank *C) is a $\mathbb{Z}$-graded vector space over $\mathbb{C}$*

$$V = \coprod_{k \in \mathbb{Z}} V_{(k)}$$

such that $\dim V_{(k)} < \infty$ *for $k \in \mathbb{Z}$, together with linear maps*

$$\begin{aligned} \varLambda : V &\mapsto (V \otimes V)[[x, x^{-1}]] \\ v &\mapsto \varLambda(x)v = \sum_{k \in \mathbb{Z}} \Delta_k(v) x^{-k-1} \end{aligned}$$

(called the coproduct),

$$c : V \mapsto \mathbb{C}$$

(called the covacuum map), and

$$\rho : V \mapsto \mathbb{C}$$

(called the co-Virasoro map),

satisfying the following 8 axioms:

1. *Positive energy:*

$$V_{(k)} = 0 \text{ for } k \text{ sufficiently small.} \tag{7.1.1}$$

2. *Left counit: For all $v \in V$*

$$(c \otimes Id_V)\varLambda(x)v = v \tag{7.1.2}$$

3. *Cocreation: For all $v \in V$*

$$(Id_V \otimes c)\varLambda(x)v \in V[[x]] \quad \text{and} \tag{7.1.3}$$

$$\lim_{x\to 0}(Id_V \otimes c)\varLambda(x)v = v. \tag{7.1.4}$$

4. *Truncation: Given $v \in V$, then $\Delta_k(v) = 0$ for k sufficiently small.*

5. *Jacobi identity:*

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)(Id_V \otimes \varLambda(x_2))\varLambda(x_1) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)(T \otimes Id_V)(Id_V \otimes \varLambda(x_1))\varLambda(x_2) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)(\varLambda(x_0) \otimes Id_V)\varLambda(x_2). \tag{7.1.5}$$

6. *Virasoro algebra: The Virasoro algebra bracket,*

$$[L(j), L(k)] = (j-k)L(j+k) + \frac{1}{12}(j^3 - j)\delta_{j,-k}C,$$

holds for $j, k \in \mathbb{Z}$, where

$$(\rho \otimes Id_V)\varLambda(x) = \sum_{k\in\mathbb{Z}} L(k)x^{k-2}. \tag{7.1.6}$$

7. *Grading: For each* $k \in \mathbb{Z}$ *and* $v \in V_{(k)}$

$$L(0)v = kv. \tag{7.1.7}$$

8. $L(1)$*-derivative:*

$$\frac{d}{dx}\Lambda(x) = (L(1) \otimes Id_V)\Lambda(x). \tag{7.1.8}$$

We denote this vertex operator coalgebra by (V, Λ, c, ρ), or sometimes by just V.

Note that x_0, x_1 and x_2 are formal commuting variables and Λ is linear so that, for example, $(Id_V \otimes \Lambda(x_1))$ acting on the coefficients of $\Lambda(x_2)v \in (V \otimes V)[[x_2, x_2^{-1}]]$ is well defined. Notice also, that when each expression is applied to any element of V, the coefficient of each monomial in the formal variables is a finite sum.

7.2 Properties of VOCs

There are a number of interesting consequences of the above axioms. Using the $L(1)$-derivative property along with the commutativity of operators with formal variables we see that for all $v' \in (V \otimes V)'$ and $v \in V$

$$\langle v', \left(e^{x_0 L(1)} \otimes Id_V\right) \Lambda(x)v\rangle = \langle v', e^{x_0 \frac{d}{dx}} \Lambda(x)v\rangle$$

Hence, by Proposition 4.5.1, we obtain

$$\left(e^{x_0 L(1)} \otimes Id_V\right) \Lambda(x) = \Lambda(x + x_0). \tag{7.2.1}$$

We get a pair of important facts by taking the residue with respect to x_0 of the Jacobi identity, (7.1.5), then composing with $\rho \otimes Id_V \otimes Id_V$; i.e.

$$\begin{aligned}(\rho \otimes Id_V \otimes Id_V)\mathrm{Res}_{x_0} x_0^{-1} &\left(\delta\left(\frac{x_1 - x_2}{x_0}\right)(Id_V \otimes \Lambda(x_2))\Lambda(x_1)\right. \\ &\left. - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)(T \otimes Id_V)(Id_V \otimes \Lambda(x_1))\Lambda(x_2)\right) \\ &= (\rho \otimes Id_V \otimes Id_V)\mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_x}\right)(\Lambda(x_0) \otimes Id_V)\Lambda(x_2)\end{aligned}$$

results in

$$\Lambda(x_2)\sum_{k\in\mathbb{Z}}L(k)x_1^{k-2}-(Id_V\otimes\sum_{k\in\mathbb{Z}}L(k)x_1^{k-2})\Lambda(x_2)$$
$$=\mathrm{Res}_{x_0}x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)(\sum_{k\in\mathbb{Z}}L(k)x_0^{k-2}\otimes Id_V)\Lambda(x_2) \quad (7.2.2)$$

by applying (7.1.6). Now applying (5.1.2) to the right-hand side of Equation (7.2.2) we have

$$\Lambda(x_2)\sum_{k\in\mathbb{Z}}L(k)x_1^{k-2}-(Id_V\otimes\sum_{k\in\mathbb{Z}}L(k)x_1^{k-2})\Lambda(x_2)$$
$$=\mathrm{Res}_{x_0}x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)(\sum_{k\in\mathbb{Z}}L(k)x_0^{k-2}\otimes Id_V)\Lambda(x_2) \quad (7.2.3)$$

Two facts which are derived from (7.2.3), called the $L(1)$-*commutation formula* and the $L(0)$-*commutation formula*, respectively, are

$$\Lambda(x_2)L(1)=(L(1)\otimes Id_V)\Lambda(x_2)+(Id_V\otimes L(1))\Lambda(x_2) \quad (7.2.4)$$

and

$$\Lambda(x_2)L(0)=(L(0)\otimes Id_V)\Lambda(x_2)+(Id_V\otimes L(0))\Lambda(x_2)+x_2(L(1)\otimes Id_V)\Lambda(x_2). \quad (7.2.5)$$

These facts are obtained from (7.2.3) by taking Res_{x_1} or $\mathrm{Res}_{x_1}x_1$, respectively. We use repeated iterations of the $L(1)$-commutation formula to obtain the formula

$$(L(1)\otimes Id_V)^m\,\Lambda(x)=\sum_{n=0}^{m}\binom{m}{n}(-Id_V\otimes L(1))^n\,\Lambda(x)L(1)^{m-n}$$

which, when used with correlation functions (i.e. $\langle v',(\cdot)v\rangle$), proves

$$\left(e^{x_0L(1)}\otimes Id_V\right)\Lambda(x)=\left(Id_V\otimes e^{-x_0L(1)}\right)\Lambda(x)e^{x_0L(1)}. \quad (7.2.6)$$

Composing $(Id_V\otimes c\otimes Id_V)$ with the Jacobi identity gives

$$x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\Lambda(x_1)=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)((Id_V\otimes c)\Lambda(x_0)\otimes Id_V)\Lambda(x_2).$$

by applying (5.1.3) and (7.1.2). We then take the residue with respect to x_1. This, along with facts (5.1.2) and (7.2.1), yields the equation

$$\left(e^{x_0 L(1)} \otimes Id_V\right) \mathcal{X}(x_2) = \left((Id_V \otimes c)\mathcal{X}(x_0) \otimes Id_V\right) \mathcal{X}(x_2).$$

We can see that the cocreation property, (7.1.3) implies that $\mathcal{X}$ is "left surjective" in the sense that its image includes all of $V \otimes \mathbb{C}$, allowing us to conclude

$$e^{x_0 L(1)} = (Id_V \otimes c)\mathcal{X}(x_0). \tag{7.2.7}$$

Notice that this proves left creation using only "left surjectivity" and the other axioms.

Another important identity is obtained by taking the residue of the Jacobi identity with respect to x_0, composing with $(Id_V \otimes Id_V \otimes c)$, then applying (7.2.7) to get

$$\begin{aligned}(Id_V \otimes e^{x_2 L(1)})\mathcal{X}(x_1) - (Id_V \otimes e^{x_1 L(1)})\mathcal{X}(x_2)& \\ = \mathcal{X}(x_1 - x_2)e^{x_2 L(1)} - \mathcal{X}(-x_2 + x_1)e^{x_2 L(1)};&\end{aligned}$$

which further simplifies using (7.2.1) and (7.2.6) to

$$T\mathcal{X}(x) = \mathcal{X}(-x)e^{xL(1)} \tag{7.2.8}$$

(here x replaces $x_2 - x_1$ to clarify the equation). Hence, we see that VOCs have a natural *skew-symmetry* just as VOAs do (cf. (5.3.11) in [H2]).

Another fundamental fact about VOCs helps us to understand the elements $\Delta_k(v) \in V \otimes V$. When $v \in V_{(k)}$, we say the v is homogeneous of *weight* k, or simply has weight k, and the grading axiom says $L(0)v = (\text{wt } v)v$. Given an element $w \in V \otimes V$, if $\left((L(0) \otimes Id_V) + (Id_V \otimes L(0))\right) w = aw$ for some $a \in \mathbb{C}$, then we say that the weight of w is equal to the scalar a, i.e., wt $w = a$. This agrees with the grading on $V \otimes V$ since for $v \in V_{(k)}$ and $w \in V_{(\ell)}$, we have $v \otimes w \in (V \otimes V)_{(k+\ell)}$ and wt $(v \otimes w) = k + \ell$.

We now take $v \in V_{(k)}$ and examine the coefficients of $\Lambda(x)v = \sum_{j\in\mathbb{Z}} \Delta_j(v)x^{-j-1}$. We apply the $L(0)$-commutation formula (7.2.5) to v and see that

$$\Lambda(x)kv = ((L(0) \otimes Id_V) + (Id_V \otimes L(0)))\,\Lambda(x)v + x(L(1) \otimes Id_V)\Lambda(x)v.$$

Using the $L(1)$-derivative property (7.1.8), and expanding the Λ operators, the equation becomes

$$\begin{aligned} k\sum_{j\in\mathbb{Z}} \Delta_j(v)x^{-j-1} = ((L(0) \otimes Id_V) + (Id_V \otimes L(0))) \sum_{j\in\mathbb{Z}} \Delta_j(v)x^{-j-1} \\ + \sum_{j\in\mathbb{Z}} (-j-1)\Delta_j(v)x^{-j-1}, \end{aligned}$$

and we conclude that

$$((L(0) \otimes Id_V) + (Id_V \otimes L(0)))\,\Delta_j(v) = (\text{wt } v + j + 1)\Delta_j(v),$$

that is

$$\text{wt } \Delta_j(v) = \text{wt } v + j + 1. \tag{7.2.9}$$

As in vertex operator algebras, it is natural to define $x_0^{L(0)}v = x_0^k v$ for $v \in V_{(k)}$ since $a^{L(0)}v = a^k v$ where $a \in \mathbb{C}^\times$. Similarly, for $a \in \mathbb{C}^\times$, $v \in V_{(k)}$ and $w \in V_{(\ell)}$ we have

$$\begin{aligned} a^{(Id_V \otimes L(0))}(v \otimes w) &= (Id_V \otimes a^{L(0)})(v \otimes w) \\ &= a^\ell (v \otimes w) \end{aligned}$$

so we define

$$\begin{aligned} x^{Id_V \otimes L(0)}(v \otimes w) &= x^\ell (v \otimes w), \\ x^{L(0) \otimes Id_V}(v \otimes w) &= x^k (v \otimes w). \end{aligned}$$

This definition has the desired consequence that

$$x_0^{L(0)\otimes Id_V + Id_V \otimes L(0)}(v\otimes w) = x_0^{\mathrm{wt}\ (v\otimes w)}(v\otimes w)$$

for $v \otimes w \in V \otimes V$. Notice that this definition of weight (i.e. the operator "wt") extends naturally to $V^{\otimes n}$ for all $n \in \mathbb{Z}_+$.

Considering $v \in V_{(k)}$, it is now clear via (7.2.9) that for each $\Delta_j(v)$,

$$\Delta_j(v) = \sum_{\ell=1}^{r_j} u_\ell \otimes w_\ell$$

where each $u_\ell \in V_{(i_j)}$, each $w_\ell \in V_{(\mathrm{wt}\ \Delta_j(v) - i_j)}$, $r_j \in \mathbb{Z}_+$, and each $i_j \in \mathbb{Z}$. By considering homogeneous vectors, it can then be seen that

$$x^{-Id_V \otimes L(0)} \mathcal{A}(x) x_0^{L(0)} = x_0^{L(0)\otimes Id_V} \mathcal{A}(x_0 x). \tag{7.2.10}$$

Thus for $a \in \mathbb{C}^\times$ we have

$$(Id_V \otimes a^{-L(0)}) \mathcal{A}(x) a^{L(0)} = (a^{L(0)} \otimes Id_V) \mathcal{A}(ax). \tag{7.2.11}$$

Understanding the properties of the representation of the Virasoro algebra is another key to understanding VOCs. (It also turns out to be invaluable in proving the sewing axiom in Section 8.2.) First, we use the cocreation axiom (7.1.3 and 7.1.4) as well as the definition of ρ, (7.1.6), to see that

$$cL(j) = 0 \in \mathrm{Hom}(V, \mathbb{C}) \text{ for } j \leq 1 \tag{7.2.12}$$

$$cL(2) = \rho. \tag{7.2.13}$$

These two facts along with the Virasoro bracket relation give us

$$\rho L(0) = 2\rho. \tag{7.2.14}$$

Armed with this fact and the $L(1)$-derivative property, we compose the $L(0)$-commutation formula with $(\rho \otimes Id_V)$ to get

$$\sum_{k\in\mathbb{Z}} L(k)x^{k-2}L(0) \ = \ 2\sum_{k\in\mathbb{Z}} L(k)x^{k-2} + L(0)\sum_{k\in\mathbb{Z}} L(k)x^{k-2} + x\frac{d}{dx}\sum_{k\in\mathbb{Z}} L(k)x^{k-2}.$$

Taking $Res_x x^{-j+1}$ and applying the outcome to any homogeneous vector $v \in V_{(k)}$, we see that

$$L(0)L(j)v = (k-j)L(j)v$$

for any $j, k \in \mathbb{Z}$. Thus, as is the case with VOAs, each operator $L(j)$ raises the weight of homogeneous vectors by $-j$. Examining homogeneous vectors, we then see that for $a \in \mathbb{C}^{\times}$, $b \in \mathbb{C}$, $k \in \mathbb{Z}$

$$a^{L(0)}e^{bL(k)} = e^{a^{-k}bL(k)}a^{L(0)}. \tag{7.2.15}$$

7.3 Rationality, associativity and commutativity

We can understand a great deal about compositions of Λ operators by investigating how to obtain them from rational functions. Let $\mathbb{C}[x_1, x_2]_S$ be the ring of rational functions obtained by inverting the products of (zero or more) elements of S, where S is the set of nonzero homogeneous linear polynomials in x_1 and x_2. Let $\iota_{1\ 2} : \mathbb{C}[x_1, x_2]_S \to \mathbb{C}[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$ be defined by mapping an element $\frac{g(x_1,x_2)}{x_1^r x_2^s (x_1-x_2)^t}$ to $\frac{g(x_1,x_2)}{x_1^r x_2^s}$ times $\frac{1}{(x_1-x_2)^t}$ expanded in nonnegative powers of x_2 where $g(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ and $r, s, t \in \mathbb{Z}$. This generalizes to finitely many formal variables, i.e. $\iota_{1\cdots n}$, and any ordering of the formal variables (cf. Section 3.1 of [FHL]).

Proposition 7.3.1. *(a)(right rationality) Let $v' \in (V^{\otimes 3})'$ and $v \in V$. Then the formal series $\langle v', (Id_V \otimes \Lambda(x_2))\Lambda(x_1)v\rangle$ is in $\mathbb{C}[x_1, x_2^{-1}][[x_1^{-1}, x_2]]$ and in fact*

$$\langle v', (Id_V \otimes \Lambda(x_2))\Lambda(x_1)v\rangle = \iota_{1\ 2} f(x_1, x_2)$$

where $f(x_1, x_2) \in \mathbb{C}[x_1, x_2]_S$ is uniquely determined and is of the form

$$f(x_1, x_2) = \frac{g(x_1, x_2)}{x_1^r x_2^s (x_1 - x_2)^t} \tag{7.3.1}$$

for some $g(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ and $r, s, t \in \mathbb{Z}$.

(b) (commutativity) It is also the case that

$$\iota_{1\ 2}^{-1}\langle v', (Id_V \otimes \Lambda(x_2))\Lambda(x_1)v\rangle = \iota_{2\ 1}^{-1}\langle v', (T \otimes Id_V)(Id_V \otimes \Lambda(x_1))\Lambda(x_2)v\rangle.$$

Proof. We may consider homogeneous vectors $v \in V_{(r)}$ and $v' \in (V^{\otimes 3})'_{(s)}$. The residue with respect to x_0 of the Jacobi identity yields

$$\langle v', ((Id_V \otimes \Lambda(x_2))\Lambda(x_1) - (T \otimes Id_V)(Id_V \otimes \Lambda(x_1))\Lambda(x_2))\, v\rangle$$
$$= \langle v', ((\Lambda(x_1 - x_2) - (\Lambda(-x_2 + x_1)) \otimes Id_V)\, \Lambda(x_2)v\rangle.$$

Here as throughout VOA theory $(x_1 - x_2)^n$ is understood to be expanded in non-negative integer powers of x_2.

By truncation, $\Delta_j(v) = 0$ for $j < M$ for some $M \in \mathbb{Z}_+$. Note that the weight of $\Delta_j(v)$ is $r + j + 1$ so the weight of the coefficients of $(\Lambda(x_0) \otimes Id_V)\Lambda(x_2)v$ will be $(r+j+1)+k+1$ where j indexes the first comultiplication and k indexes the second. But the weight of the coefficients of $((\Lambda(x_1 - x_2) - (\Lambda(-x_2 + x_1)) \otimes Id_V)\, \Lambda(x_2)v$ must match the weight of v' to be nonzero, i.e. we have $s = r + j + k + 2$. Since $(x_1 - x_2)^{-k-1} = (-x_2 + x_1)^{-k-1}$ for $k < 0$ resulting in coefficients of 0, we need $s \geq r + j + 2$ (or $j \leq s - r - 2$) to have nonzero coefficients. We conclude that j is bounded above and below. A similar argument bounds the index k above and below. Explicitly, there exists $M, N \in \mathbb{Z}_+$ such that

$$\begin{aligned}
&\langle v', ((Id_V \otimes \Lambda(x_2))\Lambda(x_1) - (T \otimes Id_V)(Id_V \otimes \Lambda(x_1))\Lambda(x_2))\, v\rangle \\
&= \sum_{j=-M}^{M} \langle v', ((\Lambda(x_1 - x_2) - (\Lambda(-x_2 + x_1)) \otimes Id_V)\, \Delta_j(v)\rangle x_2^{-j-1} \\
&= \sum_{j=-M}^{M} \sum_{k=0}^{N} a_{j,k} \left(\iota_{1\ 2}(x_1 - x_2)^{-k-1} - \iota_{2\ 1}(-x_2 + x_1)^{-k-1}\right) x_2^{-j-1} \\
&= (\iota_{1\ 2} - \iota_{2\ 1})\, \frac{g(x_1, x_2)}{x_2^{M+1}(x_1 - x_2)^{N+1}}
\end{aligned}$$

with each $a_{j,k} \in \mathbb{C}$ and $g(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ unique. Rewriting the equation as

$$\langle v', (Id_V \otimes \Lambda(x_2))\Lambda(x_1)v\rangle - \iota_{1\ 2}\frac{g(x_1, x_2)}{x_2^{M+1}(x_1 - x_2)^{N+1}}$$
$$= \langle v', (T \otimes Id_V)(Id_V \otimes \Lambda(x_1))\Lambda(x_2)v\rangle - \iota_{2\ 1}\frac{g(x_1, x_2)}{x_2^{M+1}(x_1 - x_2)^{N+1}} \quad (7.3.2)$$

allows us to examine the x_1 terms more closely. Using the truncation axiom on the left-hand side shows there are only finitely many positive powers of x_1 on either side of the equation above. By weights and the finiteness of v' on the right-hand side, there are only finitely many negative powers of x_1 as well. Further, the coefficient of each power of x_1 is a Laurent polynomial in x_2. Hence, both sides of the above equation are equal to some unique Laurent polynomial $h(x_1, x_2) \in \mathbb{C}[x_1, x_2, x_1^{-1}, x_2^{-1}]$.

This allows us to uniquely define $f(x_1, x_2) \in \mathbb{C}[x_1, x_2]_S$ by

$$f(x_1, x_2) = h(x_1, x_2) + \frac{g(x_1, x_2)}{x_2^{M+1}(x_1 - x_2)^{N+1}} = \iota_{1\,2}^{-1}\langle v', (Id_V \otimes \Lambda(x_2))\Lambda(x_1)v\rangle$$

proving part (a). Similarly, (7.3.2) shows that

$$f(x_1, x_2) = \iota_{2\,1}^{-1}\langle v', (T \otimes Id_V)(Id_V \otimes \Lambda(x_1))\Lambda(x_2)v\rangle$$

proving part (b). □

Proposition 7.3.2. *(a)(left rationality) Let $v' \in (V^{\otimes 3})'$ and $v \in V$. Then the formal series $\langle v', (\Lambda(x_0) \otimes Id_V)\Lambda(x_2)v\rangle$ is in $\mathbb{C}[x_0^{-1}, x_2][[x_0, x_2^{-1}]]$ and in fact*

$$\langle v', (\Lambda(x_0) \otimes Id_V)\Lambda(x_2)v\rangle = \iota_{2\,0}k(x_0, x_2)$$

where $k(x_0, x_2) \in \mathbb{C}[x_0, x_2]_S$ is uniquely determined and is of the form

$$k(x_0, x_2) = \frac{g(x_0, x_2)}{x_0^r x_2^s (x_0 + x_2)^t}$$

for some $g(x_0, x_2) \in \mathbb{C}[x_0, x_2]$ and $r, s, t \in \mathbb{Z}$.

(b) (associativity) It is also the case that

$$\iota_{1\,2}^{-1}\langle v', (Id_V \otimes \Lambda(x_2))\Lambda(x_1)v\rangle = \left(\iota_{2\,0}^{-1}\langle v', (\Lambda(x_0) \otimes Id_V)\Lambda(x_2)v\rangle\right)\Big|_{x_0 = x_1 - x_2}.$$

Proof. Following the proof of Proposition 7.3.1, right rationality, we let $v \in V_{(r)}$, $v' \in (V^{\otimes 3})'_{(s)}$ and take the residue with respect to x_1 of the Jacobi identity:

$$\begin{aligned}&\langle v', (Id_V \otimes \Lambda(x_2))\Lambda(x_0 + x_2)v\rangle \\ &\qquad - \langle v', (T \otimes Id_V)(Id_V \otimes (\Lambda(x_0 + x_2) - \Lambda(x_2 + x_0)))\Lambda(x_2)v\rangle \\ &\qquad\qquad = \langle v', (\Lambda(x_0) \otimes Id_V)\Lambda(x_2)v\rangle. \quad (7.3.3)\end{aligned}$$

Again as in Proposition 7.3.1, there exists $M, N \in \mathbb{Z}_+$ such that

$$\begin{aligned}
&\langle v', (T \otimes Id_V)(Id_V \otimes (\Lambda(x_0+x_2) - \Lambda(x_2+x_0)))\Lambda(x_2)v\rangle \\
&= \sum_{j=-M}^{M}\sum_{k=0}^{N} a_{j,k}\left(\iota_{0\ 2}(x_0+x_2)^{-k-1} - \iota_{2\ 0}(x_2+x_0)^{-k-1}\right)x_2^{-j-1} \\
&= (\iota_{0\ 2} - \iota_{2\ 0})\frac{g(x_0,x_2)}{x_2^{M+1}(x_0+x_2)^{N+1}}
\end{aligned}$$

with each $a_{j,k} \in \mathbb{C}$ and $g(x_0, x_2) \in \mathbb{C}[x_0, x_2]$ unique. Equation (7.3.3) may now be rewritten as

$$\begin{aligned}
\langle v', (Id_V \otimes \Lambda(x_2))\Lambda(x_0+x_2)v\rangle - \iota_{0\ 2}\frac{g(x_0,x_2)}{x_2^{M+1}(x_0+x_2)^{N+1}}& \\
= \langle v', (\Lambda(x_0) \otimes Id_V)\Lambda(x_2)v\rangle - \iota_{2\ 0}\frac{g(x_0,x_2)}{x_2^{M+1}(x_0+x_2)^{N+1}}&.
\end{aligned}$$

Again, we follow the proof of Proposition 7.3.1 and find the unique Laurent polynomial $h(x_0, x_2) \in \mathbb{C}[x_0, x_2, x_0^{-1}, x_2^{-1}]$ that is equal to both sides of the above equation. Defining $k(x_0, x_2) \in \mathbb{C}[x_0, x_2]_S$ by

$$k(x_0,x_2) = h(x_0,x_2) + \frac{g(x_0,x_2)}{x_2^{M+1}(x_0+x_2)^{N+1}} = \iota_{2\ 0}^{-1}\langle v', (\Lambda(x_0) \otimes Id_V)\Lambda(x_2)v\rangle$$

proves part (a). The fact that $f(x_1, x_2)$ from Proposition 7.3.1 is of the form (7.3.1) shows that

$$\begin{aligned}
f(x_1,x_2)|_{x_1=x_0+x_2} &= \left(\iota_{1\ 2}^{-1}\langle v', (Id_V \otimes \Lambda(x_2))\Lambda(x_1)v\rangle\right)\big|_{x_1=x_0+x_2} \\
&= \iota_{0\ 2}^{-1}\langle v', (Id_V \otimes \Lambda(x_2))\Lambda(x_0+x_2)v\rangle
\end{aligned}$$

and hence

$$f(x_0+x_2, x_2) = k(x_0, x_2). \tag{7.3.4}$$

Thus

$$\begin{aligned}
\iota_{1\ 2}^{-1}\langle v', (Id_V \otimes \Lambda(x_2))\Lambda(x_1)v\rangle &= f(x_1,x_2) \\
&= f(x_0+x_2,x_2)\big|_{x_0=x_1-x_2} \\
&= k(x_0,x_2)\big|_{x_0=x_1-x_2} \\
&= \left(\iota_{2\ 0}^{-1}\langle v', (\Lambda(x_0) \otimes Id_V)\Lambda(x_2)v\rangle\right)\big|_{x_0=x_1-x_2}.
\end{aligned}$$

□

7.4 The Jacobi identity from rationality, commutativity and associativity

Proposition 7.4.1. *The Jacobi identity may be replaced in the definition of a VOC by right and left rationality, commutativity and associativity, i.e. the claims of Propositions 7.3.1 and 7.3.2.*

Proof. Proposition 3.1.1 of [FHL] tells us that given a rational function

$$f(x_0, x_1, x_2) = \frac{g(x_0, x_1, x_2)}{x_0^r x_1^s x_2^t},$$

where g is a polynomial and $r, s, t \in \mathbb{Z}$,

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)\iota_{1\ 2}(f|_{x_0=x_1-x_2}) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\iota_{2\ 1}(f|_{x_0=x_1-x_2}) \\ = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)\iota_{2\ 0}(f|_{x_1=x_0+x_2}). \quad (7.4.1)$$

We use the rational function $f(x_1, x_2) = \frac{g(x_1,x_2)}{x_1^r x_2^s (x_1-x_2)^t}$ from Proposition 7.3.1 and define $\tilde{f}(x_0, x_1, x_2) = \frac{g(x_1,x_2)}{x_1^r x_2^s x_0^t}$. Plugging $\tilde{f}$ into equation (7.4.1), we get

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)\iota_{1\ 2}f(x_1, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\iota_{2\ 1}f(x_1, x_2) \\ = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right)\iota_{2\ 0}f(x_0 + x_2, x_2).$$

Equations (5.1.2) and (7.3.4) applied to the right-hand side of this equation gives us

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)\iota_{1\ 2}f(x_1, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\iota_{2\ 1}f(x_1, x_2) \\ = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\iota_{2\ 0}k(x_0, x_2).$$

(Using Equation (7.3.4) is valid because it comes directly from left and right rationality.)

We need only recognize that using the properties of right and left rationality, as well as commutativity gives us the Jacobi identity to finish the proof. □

CHAPTER 8

AN ISOMORPHISM BETWEEN THE CATEGORIES OF GVOCS AND VOCS

We defined geometric vertex operator coalgebras to motivate the definition of vertex operator coalgebras. We are now in a position to prove that these notions give rise to isomorphic categories. The category of GVOCs of rank C has the class of all GVOCs of rank C as its objects and its morphisms are those linear maps between GVOCs that are μ invariant. The category of VOCs of rank C has all VOCs of rank C as objects and for morphisms, linear maps between VOCs which are coproduct invariant as well as preserving the covacuum and co-Virasoro maps.

We will define a functor from GVOCs to VOCs and then a functor from VOCs to GVOCs. Finally, we will show that these two functors are inverses to each other, thus giving an isomorphism.

8.1 A map from GVOCs to VOCs

In this section we construct a map from the category of geometric vertex operator coalgebras to the category of vertex operator coalgebras. In Section 8.3, we will prove that, in fact, this map is an isomorphism.

Let (V, μ) be a geometric vertex operator coalgebra. We define a linear map $c^\mu : V \to \mathbb{C}$ by

$$c^\mu = \mu_0((1, \mathbf{0})), \tag{8.1.1}$$

a linear map $\rho^\mu : V \to \mathbb{C}$ by

$$\rho^\mu = -\frac{d}{d\varepsilon}\mu_0(1,(0,\varepsilon,0,0,\ldots))\bigg|_{\varepsilon=0}, \tag{8.1.2}$$

and a linear map $\Lambda^\mu : V \mapsto (V\otimes V)[[x,x^{-1}]]$ by

$$\mathrm{Res}_x x^n\langle v', \Lambda^\mu(x)v\rangle = \mathrm{Res}_z z^n\langle v', \mu_2((z^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0})))v\rangle,$$

for $v' \in (V\otimes V)', v\in V$ where Res_x means taking the coefficient of the x^{-1} term in the given series and Res_z means taking the residue of the function at the singularity $z=0$.

Proposition 8.1.1. *If the rank of* (V,μ) *is* C*, then the quadruple* $(V,\Lambda^\mu,c^\mu,\rho^\mu)$ *is a vertex operator coalgebra of rank* C*.*

Proof. We use the GVOC axioms to prove the VOC axioms with respect to $(V,\Lambda^\mu,c^\mu,\rho^\mu)$.

1. Positive energy: trivially verified.
2. Left counit: Let $v'\in V'$, $v\in V$. Then

$$\begin{aligned}\langle v',(c^\mu\otimes Id_V)\Lambda^\mu(x)v\rangle|_{x=z} &= \langle v',(\mu_0((1,\mathbf{0}))\otimes Id_V)\Lambda^\mu(x)v\rangle|_{x=z}\\ &= \langle v',\mu_0((1,\mathbf{0}))\ {}_1*_{-1}\ \mu_2((z^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0})))v\rangle\\ &= \langle v',\mu_1((1,\mathbf{0})\ {}_1\infty_{-1}\ (z^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0})))v\rangle\\ &= \langle v',\mu_1(\mathbf{0},(1,\mathbf{0}))v\rangle\\ &= \langle v',v\rangle.\end{aligned}$$

(The second equality uses projection maps to show convergence and the last step uses a trivial application of the grading axiom.)

3. Cocreation: Let $v' \in V'$, $v \in V$.

$$\begin{aligned}\langle v', (Id_V \otimes c^{\mu})\Lambda^{\mu}(x)v\rangle|_{x=z} &= \langle v', (Id_V \otimes \mu_0((1,\mathbf{0})))\Lambda^{\mu}(x)v\rangle|_{x=z} \\ &= \langle v', \mu_0((1,\mathbf{0}))\ {}_1*_{-2}\ \mu_2((z^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0})))v\rangle \\ &= \langle v', \mu_1((1,\mathbf{0})\ {}_1\infty_{-2}\ (z^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0})))v\rangle \\ &= \langle v', \mu_1(\mathbf{0},(1,(-z,0,0,\ldots)))v\rangle.\end{aligned}$$

By the meromorphicity axiom, $\langle v', \mu_1(\cdot)v\rangle$ is a meromorphic function on $K^*(1)$. Hence $\langle v', \mu_1(\mathbf{0},(1,(-z,0,0,\ldots)))v\rangle$ is a polynomial in z. Thus for all $v \in V$, $v' \in V'$, $\langle v', (Id_V \otimes c^{\mu})\Lambda^{\mu}(x)v\rangle \in V[x]$. We can also see that

$$\begin{aligned}\lim_{z\to 0}\langle v', \mu_1(\mathbf{0},(1,(-z,0,0,\ldots)))v\rangle &= \langle v', \mu_1(\mathbf{0},(1,\mathbf{0}))v\rangle \\ &= \langle v', v\rangle.\end{aligned}$$

4. Truncation: By the meromorphicity axiom, for any $v \in V$ there exists $N(v) \in \mathbb{Z}_+$ such that for all $v' \in (V \otimes V)'$, the power of z in $\langle v', \mu_2((z^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0})))v\rangle$ is less than $N(v)$. But

$$\langle v', \mu_2((z^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0})))v\rangle = \langle v', \Lambda^{\mu}(x)v\rangle|_{x=z};$$

and hence the number of positive powers of x in $\Lambda^{\mu}(x)v$ must be less than $N(v)$ as well.

5. Jacobi identity: The idea behind proving the Jacobi identity is to prove right and left rationality as well as commutativity and associativity. As shown in Proposition 7.4.1, this is equivalent to the Jacobi identity.

We will start by obtaining right rationality and commutativity.

$$\begin{aligned}&\langle v', (Id_V \otimes \Lambda^{\mu}(x_2))\Lambda^{\mu}(x_1)v\rangle|_{x_i=z_{-i}} \\ &\quad= \langle v', \mu_2((z_{-2}^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0})))\ {}_1*_{-2}\ \mu_2((z_{-1}^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0})))v\rangle \\ &\quad= \langle v', \mu_3((z_{-2}^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0}))\ {}_1\infty_{-2}\ (z_{-1}^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0})))v\rangle \\ &\quad= \langle v', \mu_3((z_{-1}^{-1},z_{-2}^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0}),(1,\mathbf{0})))v\rangle\end{aligned}$$

for any $z_{-1}, z_{-2} \in \mathbb{C}^\times$ for which this sewing is well defined, i.e., for $|z_{-1}| > |z_{-2}|$. Similarly,

$$\langle v', (Id_V \otimes \Lambda^\mu(x_1))\Lambda^\mu(x_2)v\rangle|_{x_i=z_{-i}} = \langle v', \mu_3((z_{-2}^{-1}, z_{-1}^{-1}; \mathbf{0}, (1,\mathbf{0}), (1,\mathbf{0}), (1,\mathbf{0})))v\rangle$$

for $|z_{-2}| > |z_{-1}|$. By the meromorphicity axiom,

$$\langle v', \mu_3((z_{-1}^{-1}, z_{-2}^{-1}; \mathbf{0}, (1,\mathbf{0}), (1,\mathbf{0}), (1,\mathbf{0})))v\rangle = \frac{g(z_{-1}, z_{-2})}{z_{-1}^r z_{-2}^s (z_{-1} - z_{-2})^t}$$

where $g(z_{-1}, z_{-2}) \in \mathbb{C}[z_{-1}, z_{-2}]$ and $r, s, t \in \mathbb{Z}$. Thus, since

$$\langle v', (Id_V \otimes \Lambda^\mu(x_2))\Lambda^\mu(x_1)v\rangle|_{x_i=z_{-i}} = \frac{g(z_{-1}, z_{-2})}{z_{-1}^r z_{-2}^s (z_{-1} - z_{-2})^t}$$

and $\langle v', (Id_V \otimes \Lambda^\mu(x_2))\Lambda^\mu(x_1)v\rangle$ has only finitely many positive powers of x_1 and hence only finitely many negative powers of x_2 by evaluating weights on homogeneous vectors,

$$\langle v', (Id_V \otimes \Lambda^\mu(x_2))\Lambda^\mu(x_1)v\rangle = \iota_{1\,2} \frac{g(x_1, x_2)}{x_1^r x_2^s (x_1 - x_2)^t}, \tag{8.1.3}$$

i.e., $(V, \Lambda^\mu, c^\mu, \rho^\mu)$ satisfies right rationality. By the permutation axiom,

$$\begin{aligned}&\langle v', \mu_3((z_{-1}^{-1}, z_{-2}^{-1}; \mathbf{0}, (1,\mathbf{0}), (1,\mathbf{0}), (1,\mathbf{0})))v\rangle \\ &\qquad = \langle v', (T \otimes Id_V)\mu_3((z_{-2}^{-1}, z_{-1}^{-1}; \mathbf{0}, (1,\mathbf{0}), (1,\mathbf{0}), (1,\mathbf{0})))v\rangle;\end{aligned}$$

hence

$$\begin{aligned}\iota_{1\,2}^{-1}\langle v', (Id_V \otimes \Lambda^\mu(x_2))\Lambda^\mu(x_1)v\rangle &= \frac{g(x_1, x_2)}{x_1^r x_2^s (x_1 - x_2)^t} \\ &= \iota_{2\,1}^{-1}\langle v', (T \otimes Id_V)(Id_V \otimes \Lambda^\mu(x_1))\Lambda^\mu(x_2)v\rangle\end{aligned}$$

proving commutativity. Now working on toward left rationality and associativity, we use the same technique to argue that

$$\begin{aligned}&\langle v', (\Lambda^\mu(x_0) \otimes Id_V)\Lambda^\mu(x_2)v\rangle|_{x_0=(z_{-1}-z_{-2}), x_2=z_{-2}} \\ &\quad = \langle v', \mu_2(((z_{-1} - z_{-2})^{-1}; \mathbf{0}, (1,\mathbf{0}), (1,\mathbf{0})))\ {}_1*_{-2}\ \mu_2((z_{-2}^{-1}; \mathbf{0}, (1,\mathbf{0}), (1,\mathbf{0})))v\rangle \\ &\qquad = \langle v', \mu_3((z_{-1}, z_{-2}; \mathbf{0}, (1,\mathbf{0}), (1,\mathbf{0}), (1,\mathbf{0})))v\rangle.\end{aligned}$$

for $|z_{-2}| > |z_{-1} - z_{-2}|$. Recall that the right-hand side of this equation is equal to

$$\frac{g(z_{-1}, z_{-2})}{z_{-1}^r z_{-2}^s (z_{-1} - z_{-2})^t} = \frac{h(z_{-1} - z_{-2}, z_{-2})}{((z_{-1} - z_{-2}) + z_{-2})^r z_{-2}^s (z_{-1} - z_{-2})^t}$$

for some $h(z_{-1} - z_{-2}, z_{-2}) \in \mathbb{C}[z_{-1}, z_{-2}]$. Thus, as in (8.1.3),

$$\langle v', (Id_V(\Lambda^\mu(x_0) \otimes Id_V)\Lambda^\mu(x_2)v\rangle = \iota_{2\ 0} \frac{h(x_0, x_2)}{(x_0 + x_2)^r x_2^s x_0^t},$$

that is, left rationality holds. In addition, we see that

$$\begin{aligned} &\left(\iota_{2\ 0}^{-1}\langle v', (Id_V(\Lambda^\mu(x_0) \otimes Id_V)\Lambda^\mu(x_2)v\rangle\right)\Big|_{x_0 = x_1 - x_2} \\ &\qquad = \frac{h(x_{-1} - x_{-2}, x_{-2})}{((x_{-1} - x_{-2}) + x_{-2})^r x_{-2}^s (x_{-1} - x_{-2})^t} \\ &\qquad = \iota_{1\ 2}^{-1}\langle v', (Id_V \otimes \Lambda^\mu(x_2))\Lambda^\mu(x_1)v\rangle, \end{aligned}$$

which is, of course, associativity.

6. Virasoro algebra: In order to examine the Virasoro algebra structure of the $L(k)$ operators, we must first determine what

$$(\rho \otimes Id_V)\Lambda(x) = \sum_{k \in \mathbb{Z}} L(k) x^{k-2}$$

forces the $L(k)$ operators to be. We will make use of the linear functionals $\mathcal{L}_I(z)$ described in Section 4.6. If we choose $v' \in V'$ and $v \in V$ then apply $\mathcal{L}_I(z)$ to $\langle v', \mu_1(\cdot)v\rangle$, there are two different ways to interpret the result. On the one hand,

$$\begin{aligned} &\mathcal{L}_I(z)\langle v', \mu_1(\cdot)v\rangle \\ &\quad = \left(\frac{d}{d\varepsilon}\langle v', \mu_1(((1, (0, -\varepsilon, 0, 0, \ldots)))\ {}_1\infty_{-1}\ (z^{-1}; \mathbf{0}, (1, \mathbf{0}), (1, \mathbf{0})))v\rangle\right)\Bigg|_{\varepsilon = 0} \\ &\quad = \left(\frac{d}{d\varepsilon}\langle v', \mu_0(\mathbf{0}, (1, (0, -\varepsilon, 0, 0, \ldots)))\ {}_1*_{-1}\ \Lambda^\mu(x))v\rangle\right)\Bigg|_{\varepsilon = 0, x = z} \\ &\quad = \langle v', \frac{-d}{d\varepsilon}\mu_0(\mathbf{0}, (1, (0, \varepsilon, 0, 0, \ldots)))\Bigg|_{\varepsilon = 0}\ {}_1*_{-1}\ \Lambda^\mu(x))v\rangle\Big|_{x = z} \\ &\quad = \langle v', (\rho^\mu \otimes Id_V)\Lambda^\mu(x)v\rangle\Big|_{x = z} \end{aligned}$$

On the other hand, viewing $\mathcal{L}_I(z)$ as an element of the tangent space of $K^*(1)$ at $(\mathbf{0}, (1, \mathbf{0}))$, Lemma 4.6.2 tells us that

$$
\begin{aligned}
&\mathcal{L}_I(z)\langle v', \mu_1(\cdot)v\rangle \\
&\quad = \sum_{k\in\mathbb{Z}_+} z^{-k-2} \frac{-\partial}{\partial A_k^{(-1)}} \langle v', \mu_1((a_0^{(1)}, A^{(1)}), A^{(-1)})v\rangle\Big|_{A^{(-1)}, A^{(1)}=\mathbf{0},\ a_0^{(1)}=1} \\
&\qquad + z^{-2} \frac{-\partial}{\partial a_0^{(1)}} \langle v', \mu_1((a_0^{(1)}, A^{(1)}), A^{(-1)})v\rangle\Big|_{A^{(-1)}, A^{(1)}=\mathbf{0},\ a_0^{(1)}=1} \\
&\qquad + \sum_{k\in\mathbb{Z}_+} z^{k-2} \frac{-\partial}{\partial A_k^{(1)}} \langle v', \mu_1((a_0^{(1)}, A^{(1)}), A^{(-1)})v\rangle\Big|_{A^{(-1)}, A^{(1)}=\mathbf{0},\ a_0^{(1)}=1} \\
&\quad = \sum_{k\in\mathbb{Z}_+} \langle v', z^{-k-2} L(-k)v\rangle + \langle v', z^{-2}L(0)v\rangle + \sum_{k\in\mathbb{Z}_+} \langle v', z^{k-2}L(k)v\rangle \\
&\quad = \langle v', \sum_{k\in\mathbb{Z}} L(k)x^{k-2}v\rangle\Big|_{x=z}
\end{aligned}
$$

where the $L(k)$'s are defined by

$$
\begin{aligned}
\langle v', L(k)v\rangle &= -\frac{\partial}{\partial A_k^{(1)}} \langle v', \mu_1((a_0^{(1)}, A^{(1)}), A^{(-1)})v\rangle\Big|_{A^{(-1)}, A^{(1)}=\mathbf{0},\ a_0^{(1)}=1}, \\
\langle v', L(0)v\rangle &= -\frac{\partial}{\partial a_0^{(1)}} \langle v', \mu_1((a_0^{(1)}, A^{(1)}), A^{(-1)})v\rangle\Big|_{A^{(-1)}, A^{(1)}=\mathbf{0},\ a_0^{(1)}=1}, \\
\langle v', L(-k)v\rangle &= -\frac{\partial}{\partial A_k^{(-1)}} \langle v', \mu_1((a_0^{(1)}, A^{(1)}), A^{(-1)})v\rangle\Big|_{A^{(-1)}, A^{(1)}=\mathbf{0},\ a_0^{(1)}=1}
\end{aligned}
$$

for $k > 0$.

This geometric definition of the $L(k)$ operators is identical to the definition of the $L(k)$ operators in the vertex operator algebra setting of [H2]. Thus we may claim the proof from vertex operator algebras ([H2], (5.4.26)) that these operators satisfy the Virasoro relations.

7. Grading: If we assume that $k \in \mathbb{Z}$, $v \in V_{(k)}$, $v' \in V'$, then observe that by the

definition of $L(0)$ and the GVOC grading axiom

$$\begin{aligned}\langle v', L(0)v\rangle &= \left(-\frac{\partial}{\partial a_0^{(1)}}\langle v', \mu_1(\mathbf{0},(a_0,\mathbf{0}))v\rangle\right)\Bigg|_{a_0=1} \\ &= \left(-\frac{\partial}{\partial a_0^{(1)}}(a_0^{(1)})^{-k}\langle v', v\rangle\right)\Bigg|_{a_0=1} \\ &= \langle v', kv\rangle.\end{aligned}$$

8. $L(1)$-Derivative: Let $v' \in (V\otimes V)'$, $v \in V_{(k)}$.

$$\begin{aligned}&\langle v', (L(1)\otimes Id_V)\Lambda^{\mu}(x)v\rangle|_{x=z} \\ &= \left(\frac{-\partial}{\partial z_0}\langle v', (\mu_1(\mathbf{0},(1,(z_0,0,0,\ldots)))\otimes Id_V)\Lambda^{\mu}(x)v\rangle\right)\Bigg|_{x=z, z_0=0} \\ &= \left(\frac{-\partial}{\partial z_0}\langle v', \mu_1(\mathbf{0},(1,(z_0,0,0,\ldots)))_1 *_{-1} \mu_2((z^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0})))v\rangle\right)\Bigg|_{z_0=0} \\ &= \left(\frac{-\partial}{\partial z_0}\langle v', \mu_2(\mathbf{0},(1,(z_0,0,0,\ldots)))_1\infty_{-1}(z^{-1};\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0})))v\rangle\right)\Bigg|_{z_0=0} \\ &= \left(\frac{-\partial}{\partial z_0}\langle v', \mu_2(z-z_0;\mathbf{0},(1,\mathbf{0}),(1,\mathbf{0}))v\rangle\right)\Bigg|_{z_0=0} \\ &= \left(\frac{-\partial}{\partial z_0}\left(\langle v', \Lambda^{\mu}(x)v\rangle|_{x=z-z_0}\right)\right)\Bigg|_{z_0=0} \\ &= \left(\langle v', \frac{-\partial}{\partial x}\Lambda^{\mu}(x)v\rangle\Bigg|_{x=z-z_0}\frac{\partial}{\partial z_0}(z-z_0)\right)\Bigg|_{z_0=0} \\ &= \langle v', \frac{\partial}{\partial x}\Lambda^{\mu}(x)v\rangle\Bigg|_{x=z}\end{aligned}$$

This completes the proof that all the axioms hold for $(V, \Lambda^{\mu}, c^{\mu}, \rho^{\mu})$. □

8.2 A map from VOCs to GVOCs

We now construct a map from the category of vertex operator coalgebras to the category of geometric vertex operator coalgebras, which extends naturally to a functor between categories.

Let $(V, \mathit{\Lambda}, c, \rho)$ be a VOC. Let $\mu_n^{\mathit{\Lambda}} : K^*(n) \mapsto \mathcal{H}_V(1, n)$ be defined by

$$\begin{aligned}
&\langle v', \mu_n^{\mathit{\Lambda}}(z_{-n+1}^{-1}, \ldots, z_{-1}^{-1}; A^{(-n)}, (a_0^{(-n+1)}, A^{(-n+1)}), \ldots, (a_0^{(-1)}, A^{(-1)}), (a_0^{(1)}, A^{(1)}))v\rangle \\
&= \iota_{1\cdots n-1}^{-1}\langle v', \Big((a_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-1)} L(-j)} \otimes \cdots \otimes (a_0^{(-n+1)})^{-L(0)} \\
&e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-n+1)} L(-j)} \otimes e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-n)} L(-j)}\Big) (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes \mathit{\Lambda}(x_{n-1})) \cdots \\
&(Id_V \otimes Id_V \otimes \mathit{\Lambda}(x_3))(Id_V \otimes \mathit{\Lambda}(x_2))\mathit{\Lambda}(x_1)\, e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)} (a_0^{(1)})^{-L(0)} v\rangle\Big|_{x_i = z_{-i}}
\end{aligned}$$

for $n \in \mathbb{Z}_+$ and

$$\mu_0^{\mathit{\Lambda}}((1, A^{(1)}))v = c e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)} v$$

for all $v \in V$, $v' \in (V^{\otimes n})'$.

At this point we need a lemma that gives us some information about absolute convergence when two sewings are composed. In and of itself this lemma says little because of its strong conditions, but it will be necessary in the general proof that $(V, \{\mu_n^{\mathit{\Lambda}}\}_{n\in\mathbb{Z}})$ is a GVOC.

Lemma 8.2.1. *Let $\ell \in \mathbb{N}$ and $m, n \in \mathbb{Z}_+$. Choose $Q_1 \in K^*(\ell)$, $Q_2 \in K^*(m)$, $Q_3 \in K^*(n)$ and integers $1 \le i \le m$ and $1 \le j \le n$ such that the sewing $(Q_1\ {}_1\infty_{-i}\ Q_2)\ {}_1\infty_{-j} Q_3$ exists. If, for all $v' \in (V^{\otimes \ell+m+n-2})'$, $v \in V$,*

$$\langle v', \mu_{\ell+m-1}^{\mathit{\Lambda}}(Q_1\ {}_1\infty_{-i}\ Q_2)v\rangle = \langle v', (\mu_\ell^{\mathit{\Lambda}}(Q_1)\ {}_1*_{-i}\ \mu_m^{\mathit{\Lambda}}(Q_2))v\rangle e^{\Gamma_1 C}$$

and

$$\langle v', \mu_{\ell+m+n-2}^{\mathit{\Lambda}}((Q_1\ {}_1\infty_{-i}\ Q_2)\ {}_1\infty_{-j}\ Q_3)v\rangle = \\ \langle v', (\mu_{\ell+m-1}^{\mathit{\Lambda}}(Q_1\ {}_1\infty_{-i}\ Q_2)\ {}_1*_{-j}\ \mu_n^{\mathit{\Lambda}}(Q_3))v\rangle e^{\Gamma_2 C}$$

for Γ_1, Γ_2 appropriately chosen, then we have

$$\langle v', \mu_{\ell+m+n-2}^{\mathit{\Lambda}}((Q_1\ {}_1\infty_{-i}\ Q_2)\ {}_1\infty_{-j}\ Q_3)v\rangle = \\ \langle v', ((\mu_\ell^{\mathit{\Lambda}}(Q_1)\ {}_1*_{-i}\ \mu_m^{\mathit{\Lambda}}(Q_2))\ {}_1*_{-j}\ \mu_n^{\mathit{\Lambda}}(Q_3))v\rangle e^{(\Gamma_1+\Gamma_2)C} \quad (8.2.1)$$

and, in particular, the right-hand side does exist.

Similarly, let $\ell, m \in \mathbb{N}$ *and* $n \in \mathbb{Z}_+$ *such that* $m+n>1$, $1 \le i \le m+n-1$, $1 \le j \le n$, *choose* $Q_1 \in K^*(\ell)$, $Q_2 \in K^*(m)$, $Q_3 \in K^*(n)$ *so that that* $Q_1 \;{}_1\infty_{-i}\; (Q_2 \;{}_1\infty_{-j} Q_3)$ *exists. If, for all* $v' \in (V^{\otimes \ell+m+n-2})'$, $v \in V$,

$$\langle v', \mu^{\Lambda}_{m+n-1}(Q_2 \;{}_1\infty_{-j}\; Q_3)v\rangle = \langle v', (\mu^{\Lambda}_m(Q_2) \;{}_1*_{-j}\; \mu^{\Lambda}_n(Q_3))v\rangle e^{\Gamma_1 C}$$

and

$$\begin{aligned}\langle v', \mu^{\Lambda}_{\ell+m+n-2}&(Q_1 \;{}_1\infty_{-i}\; (Q_2 \;{}_1\infty_{-j} Q_3))v\rangle \\ &= \langle v', (\mu^{\Lambda}_\ell(Q_1) \;{}_1*_{-i}\; \mu^{\Lambda}_{m+n-1}(Q_2 \;{}_1\infty_{-j}\; Q_3))v\rangle e^{\Gamma_2 C}\end{aligned}$$

for Γ_1, Γ_2 *appropriately chosen, then*

$$\begin{aligned}\langle v', \mu^{\Lambda}_{\ell+m+n-2}&(Q_1 \;{}_1\infty_{-i}\; (Q_2 \;{}_1\infty_{-j} Q_3))v\rangle \\ &= \langle v', (\mu^{\Lambda}_\ell(Q_1) \;{}_1*_{-i}\; (\mu^{\Lambda}_m(Q_2) \;{}_1*_{-j}\; \mu^{\Lambda}_n(Q_3)))v\rangle e^{(\Gamma_1+\Gamma_2)C}\end{aligned}$$

and, in particular, the right-hand side does exist.

Proof. This result uses the Fischer-Grauert Theorem (Theorem 3.4.3 in [H2]) and is essentially the double absolute convergence result given in part (5e) of Proposition 5.4.1 in [H2]. (It also follows Proposition 7.1 in [B]). To prove the first half of the lemma, let $t_1, t_2 \in \mathbb{C}^\times$ and then multiply the local coordinate maps at the $-i$-th puncture of Q_2 by t_1 and at the $-j$-th puncture of Q_3 by t_2 to obtain $Q_2(t_1) \in K^*(m)$ and $Q_3(t_2) \in K^*(n)$, respectively. There exists a neighborhood of $(1,1) \in \mathbb{C}^\times \times \mathbb{C}^\times$ such that the sewing still exists. Using the definitions necessary to expand both sides of (8.2.1), and (7.2.11), we can observe that substituting $Q_2(t_1)$ and $Q_3(t_2)$ for Q_2 and Q_3 on the left-hand side of (8.2.1) shows that both sides are equal when viewed as formal power series in t_1 and t_2. The left-hand side is doubly absolutely convergent in the above neighborhood by the Fischer-Grauert Theorem, and hence, the right-hand side is doubly absolutely convergent at $t_1 = t_2 = 1$.

The argument for the second half of the lemma is similar. □

We now move to the main proposition of this section.

Proposition 8.2.2. *The pair* $(V, \{\mu_n^{\Lambda}\}_{n\in\mathbb{N}})$ *is a geometric vertex operator coalgebra of rank* C*, where* C *is the rank of the VOC* (V, Λ, c, ρ)*.*

Proof. We use the axioms of a VOC to obtain the axioms of a GVOC:

1. Positive Energy: trivially verified.

2. Grading: Let $v' \in V'$ and $v \in V_{(k)}$. Then by the VOC grading axiom, (7.1.7),

$$\begin{aligned}\langle v', \mu_1^{\Lambda}(\mathbf{0}, (a, \mathbf{0}))v\rangle &= \langle v', a^{-L(0)}v\rangle \\ &= a^{-k}\langle v', v\rangle.\end{aligned}$$

3. Meromorphicity: From right rationality (Proposition 7.3.1) and the definition of μ_n^{Λ}, we see that the map $Q \to \langle v', \mu_n^{\Lambda}(Q)v\rangle$ is meromorphic on $K(n)$. Fixing $v \in V$, we must show that for some $N(v) \in \mathbb{Z}_+$ the degree of z_{-i} is less than $N(v)$ in $\langle v', \mu_n^{\Lambda}(Q)v\rangle$ for any $1 \leq i \leq n-1$ and $v' \in (V^{\otimes n})'$. We may assume $Q = (z_{-n+1}^{-1}, \ldots, z_{-1}^{-1}; \mathbf{0}, (1, \mathbf{0}), \ldots, (1, \mathbf{0}))$.

Considering repeated application of commutativity (Proposition 7.3.1), we see that

$$\begin{aligned}
&\langle v', \mu_n^{\Lambda}(Q)v\rangle \\
&\qquad = \iota_{1\cdots n-1}^{-1}\langle v', (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes \Lambda(x_{n-1}))\cdots(Id_V \otimes \Lambda(x_2))\Lambda(x_1)\; v\rangle\big|_{x_\ell = z_{-\ell}} \\
&= \iota_{1\cdots i-2\;i\;i-1\;i+1\cdots n-1}^{-1}\langle v', (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes \Lambda(x_{n-1}))\cdots \\
&(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{i} \otimes \Lambda(x_{i+1}))(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{i-2} \otimes (T \otimes Id_V)(Id_V \otimes \Lambda(x_{i-1}))\Lambda(x_i)) \\
&\qquad\qquad (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{i-3} \otimes \Lambda(x_{i-2}))\cdots(Id_V \otimes \Lambda(x_2))\Lambda(x_1)\; v\rangle\big|_{x_\ell = z_{-\ell}} \\
&= \iota_{i\;1\cdots\hat{i}\cdots n-1}^{-1}\langle v', (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes \Lambda(x_{n-1}))\cdots \\
&\qquad (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{i} \otimes \Lambda(x_{i+1}))(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{i-2} \otimes \\
&\qquad (T \otimes Id_V)(Id_V \otimes \Lambda(x_{i-1}))\cdots(T \otimes Id_V)(Id_V \otimes \Lambda(x_1))\Lambda(x_i)\; v\rangle\big|_{x_\ell = z_{-\ell}}.
\end{aligned}$$

By the truncation axiom, there exists $N(v) \in \mathbb{Z}_+$ such that the term $\Lambda(x_i)v$ has less than $N(v)$ positive powers of x_i. Thus the expression $\langle v', \mu_n^{\Lambda}(Q)v\rangle$ has less than $N(v)$ positive powers of z_{-i}.

4. Permutation: Let

$$Q = (z_{-n+1}^{-1}, \ldots, z_{-1}^{-1}; A^{(-n)}, (a_0^{(-n+1)}, A^{(-n+1)}), \ldots, (a_0^{(-1)}, A^{(-1)}), \\ (a_0^{(1)}, A^{(1)})) \in K^*(n).$$

S_n is generated by the transpositions $\sigma = (i\ i+1)$ for $i = 1, 2, \ldots, n-1$ so it suffices to show that the axiom holds for these. If $i < n-1$, we see from Equation (4.3.1) that

$$\begin{aligned}
&\langle v', \mu_n^{\Lambda}(\sigma(Q))v\rangle \\
&= \langle v', \mu_n^{\Lambda}(z_{-n+1}^{-1}, \ldots, z_{-i-2}^{-1}, z_{-i}^{-1}, z_{-i-1}^{-1}, z_{-i+1}^{-1}, \ldots, z_{-1}^{-1}; A^{(-n)}, (a_0^{(-n+1)}, A^{(-n+1)}), \\
&\quad \ldots, (a_0^{(-i-2)}, A^{(-i-2)}), (a_0^{(-i)}, A^{(-i)}), (a_0^{(-i-1)}, A^{(-i-1)}), \\
&\quad (a_0^{(-i+1)}, A^{(-i+1)}), \ldots, (a_0^{(-1)}, A^{(-1)}), (a_0^{(1)}, A^{(1)}))v\rangle \\
&= \iota^{-1}_{1\cdots i-1\ i+1\ i\ i+2\cdots n-1}\langle v', \Big((a_0^{(-1)})^{-L(0)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-1)} L(-k)} \otimes \cdots \\
&\quad \otimes (a_0^{(-i+1)})^{-L(0)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-i+1)} L(-k)} \otimes (a_0^{(-i-1)})^{-L(0)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-i-1)} L(-k)} \\
&\quad \otimes (a_0^{(-i)})^{-L(0)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-i)} L(-k)} \otimes (a_0^{(-i-2)})^{-L(0)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-i-2)} L(-k)} \\
&\quad \otimes \cdots \otimes (a_0^{(-n+1)})^{-L(0)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-n+1)} L(-k)} \otimes e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-n)} L(-k)} \Big) \\
&\quad (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes \Lambda(x_{n-1})) \cdots (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{i+1} \otimes \Lambda(x_{i+2})) \\
&\quad (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{i} \otimes \Lambda(x_i))(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{i-1} \otimes \Lambda(x_{i+1}))(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{i-2} \otimes \\
&\quad \Lambda(x_{i-1})) \cdots (Id_V \otimes \Lambda(x_2))\Lambda(x_1)\ e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(1)} L(k)} (a_0^{(1)})^{-L(0)} v\rangle \Big|_{x_\ell = z_{-\ell}}. \qquad (8.2.2)
\end{aligned}$$

Next, by commutativity (Proposition 7.3.1), we see that the right-hand side of (8.2.2) is equal to

$$
\begin{aligned}
&\iota_{1\cdots n-1}^{-1}\langle v', \Big((a_0^{(-1)})^{-L(0)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-1)}L(-k)}\otimes\cdots\\
&\otimes (a_0^{(-i+1)})^{-L(0)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-i+1)}L(-k)}\otimes (a_0^{(-i-1)})^{-L(0)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-i-1)}L(-k)}\\
&\otimes (a_0^{(-i)})^{-L(0)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-i)}L(-k)}\otimes (a_0^{(-i-2)})^{-L(0)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-i-2)}L(-k)}\\
&\otimes\cdots\otimes (a_0^{(-n+1)})^{-L(0)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-n+1)}L(-k)}\otimes e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-n)}L(-k)}\Big)\\
&(\underbrace{Id_V\otimes\cdots\otimes Id_V}_{n-2}\otimes\Lambda(x_{n-1}))\cdots(\underbrace{Id_V\otimes\cdots\otimes Id_V}_{i+1}\otimes\Lambda(x_{i+2}))\\
&(\underbrace{Id_V\otimes\cdots\otimes Id_V}_{i-1}\otimes(T\otimes Id_V)(Id_V\otimes\Lambda(x_{i+1}))(\underbrace{Id_V\otimes\cdots\otimes Id_V}_{i-1}\otimes\Lambda(x_i))\\
&(\underbrace{Id_V\otimes\cdots\otimes Id_V}_{i-2}\otimes\Lambda(x_{i-1}))\cdots(Id_V\otimes\Lambda(x_2))\Lambda(x_1)\\
&e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(1)}L(k)}(a_0^{(1)})^{-L(0)}v\rangle\Big|_{x_\ell=z_{-\ell}}
\end{aligned}
$$

$$
\begin{aligned}
=&\iota_{1\cdots n-1}^{-1}\langle v', (\underbrace{Id_V\otimes\cdots\otimes Id_V}_{i-1}\otimes T\otimes\underbrace{Id_V\otimes\cdots\otimes Id_V}_{n-i-1})\Big((a_0^{(-1)})^{-L(0)}\\
&e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-1)}L(-k)}\otimes\cdots\otimes\cdots\otimes(a_0^{(-n+1)})^{-L(0)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-n+1)}L(-k)}\otimes\\
&e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-n)}L(-k)}\Big)(\underbrace{Id_V\otimes\cdots\otimes Id_V}_{n-2}\otimes\Lambda(x_{n-1}))\cdots(Id_V\otimes\Lambda(x_2))\Lambda(x_1)\\
&e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(1)}L(k)}(a_0^{(1)})^{-L(0)}v\rangle\Big|_{x_\ell=z_{-\ell}}\\
=&\langle v', \sigma(\mu_n^{\Lambda}(Q))v\rangle.
\end{aligned}
$$

If $i=n-1$, however, the argument requires not commutativity but the algebraic facts (7.2.1), (7.2.6) and (7.2.8). We argue as follows.

$$\begin{aligned}
&\langle v', \sigma(\mu_n^{\Lambda}(Q))v\rangle \\
&= \iota_{1\cdots n-1}^{-1}\langle v', (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes T)\Big((a_0^{(-1)})^{-L(0)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-1)}L(-k)}\otimes \\
&\quad \cdots \otimes (a_0^{(-n+1)})^{-L(0)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-n+1)}L(-k)} \otimes e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-n)}L(-k)}\Big) \\
&\quad (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes \Lambda(x_{n-1}))\cdots(Id_V\otimes \Lambda(x_2))\Lambda(x_1) \\
&\quad e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(1)}L(k)}(a_0^{(1)})^{-L(0)}v\rangle\Big|_{x_\ell = z_{-\ell}}
\end{aligned}$$

$$\begin{aligned}
&= \iota_{1\cdots n-1}^{-1}\langle v', \Big((a_0^{(-1)})^{-L(0)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-1)}L(-k)} \otimes \cdots \otimes (a_0^{(-n+2)})^{-L(0)} \\
&\quad e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-n+2)}L(-k)} \otimes e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-n)}L(-k)} \otimes (a_0^{(-n+1)})^{-L(0)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-n+1)}L(-k)}\Big) \\
&\quad (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes T\Lambda(x_{n-1}))(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-3} \otimes \Lambda(x_{n-2}))\cdots \\
&\quad (Id_V\otimes \Lambda(x_2))\Lambda(x_1)\ e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(1)}L(k)}(a_0^{(1)})^{-L(0)}v\rangle\Big|_{x_\ell = z_{-\ell}}
\end{aligned}$$

$$\begin{aligned}
&= \iota_{1\cdots n-1}^{-1}\langle v', \Big((a_0^{(-1)})^{-L(0)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-1)}L(-k)} \otimes \cdots \otimes (a_0^{(-n+2)})^{-L(0)} \\
&\quad e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-n+2)}L(-k)} \otimes e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-n)}L(-k)} \otimes (a_0^{(-n+1)})^{-L(0)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-n+1)}L(-k)}\Big) \\
&\quad (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes \Lambda(-x_{n-1})e^{x_{n-1}L(1)})(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-3} \otimes \Lambda(x_{n-2}))\cdots \\
&\quad (Id_V\otimes \Lambda(x_2))\Lambda(x_1)\ e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(1)}L(k)}(a_0^{(1)})^{-L(0)}v\rangle\Big|_{x_\ell = z_{-\ell}}
\end{aligned}$$

$$\begin{aligned}
&= \iota_{1\cdots n-1}^{-1}\langle v', \Big((a_0^{(-1)})^{-L(0)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-1)}L(-k)} \otimes \cdots \otimes (a_0^{(-n+2)})^{-L(0)} \\
&\quad e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-n+2)}L(-k)} \otimes e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-n)}L(-k)} \otimes (a_0^{(-n+1)})^{-L(0)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(-n+1)}L(-k)}\Big) \\
&\quad (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes \Lambda(-x_{n-1}))(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-3} \otimes \Lambda(x_{n-2}-x_{n-1}))\cdots \\
&\quad (Id_V\otimes \Lambda(x_2-x_{n-1}))\Lambda(x_1-x_{n-1})\ e^{x_{n-1}L(1)}e^{-\sum_{k\in\mathbb{Z}_+}A_k^{(1)}L(k)}(a_0^{(1)})^{-L(0)}v\rangle\Big|_{x_\ell = z_{-\ell}}
\end{aligned}$$

$$= \iota^{-1}_{1\cdots n-1}\langle v', \Big((a_0^{(-1)})^{-L(0)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-1)}L(-k)} \otimes \cdots \otimes (a_0^{(-n+2)})^{-L(0)}$$

$$e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-n+2)}L(-k)} \otimes e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-n)}L(-k)} \otimes (a_0^{(-n+1)})^{-L(0)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-n+1)}L(-k)} \Big)$$

$$(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes \varLambda(x_{n-1}))(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-3} \otimes \varLambda(x_{n-2}))\cdots$$

$$(Id_V \otimes \varLambda(x_2))\varLambda(x_1) e^{x_{n-1}L(1)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(1)}L(k)}$$

$$(a_0^{(1)})^{-L(0)} v\rangle\Big|_{x_{n-1}=-z_{-n+1},\ x_{n-2}=z_{-n+2}-z_{-n+1},\ldots,\ x_1=z_{-1}-z_{-n+1}}$$

We now move the $(a_0^{(-n+1)})^{-L(0)}$ term into an appropriate position using (7.2.11). We get

$$\langle v', \sigma(\mu_n^{\varLambda}(Q))v\rangle$$

$$= \iota^{-1}_{1\cdots n-1}\langle v', \Big((a_0^{(-1)})^{-L(0)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-1)}L(-k)} \otimes \cdots \otimes (a_0^{(-n+2)})^{-L(0)}$$

$$e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-n+2)}L(-k)} \otimes e^{-\sum_{k\in\mathbb{Z}_+} A_n^{(-k)}L(-k)} \otimes (a_0^{(-n+1)})^{-L(0)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-n+1)}L(-k)}$$

$$(a_0^{(-n+1)})^{L(0)} \Big) (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes (Id_V \otimes (a_0^{(-n+1)})^{-L(0)})\varLambda(x_{n-1}))$$

$$(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-3} \otimes \varLambda(x_{n-2}))\cdots(Id_V \otimes \varLambda(x_2))\varLambda(x_1) e^{x_{n-1}L(1)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(1)}L(k)}$$

$$(a_0^{(1)})^{-L(0)} v\rangle\Big|_{x_{n-1}=-z_{-n+1},\ x_{n-2}=z_{-n+2}-z_{-n+1},\ldots,\ x_1=z_{-1}-z_{-n+1}}$$

$$= \iota^{-1}_{1\cdots n-1}\langle v', \Big((a_0^{(-1)})^{-L(0)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-1)}L(-k)} (a_0^{(-n+1)})^{L(0)} \otimes \cdots \otimes$$

$$(a_0^{(-n+2)})^{-L(0)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-n+2)}L(-k)} (a_0^{(-n+1)})^{L(0)} \otimes e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-n)}L(-k)}$$

$$(a_0^{(-n+1)})^{L(0)} \otimes (a_0^{(-n+1)})^{-L(0)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(-n+1)}L(-k)} (a_0^{(-n+1)})^{L(0)} \Big)$$

$$(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes \varLambda(a_0^{(-n+1)}x_{n-1}))(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-3} \otimes \varLambda(a_0^{(-n+1)}x_{n-2}))\cdots$$

$$(Id_V \otimes \varLambda(a_0^{(-n+1)}x_2))\varLambda(a_0^{(-n+1)}x_1)(a_0^{(-n+1)})^{-L(0)} e^{x_{n-1}L(1)} e^{-\sum_{k\in\mathbb{Z}_+} A_k^{(1)}L(k)}$$

$$(a_0^{(1)})^{-L(0)} v\rangle\Big|_{x_{n-1}=-z_{-n+1},\ x_{n-2}=z_{-n+2}-z_{-n+1},\ldots,\ x_1=z_{-1}-z_{-n+1}}$$

$$
\begin{aligned}
&= \iota_{1\cdots n-1}^{-1}\langle v', \Bigg(\left(\frac{a_0^{(-1)}}{a_0^{(-n+1)}}\right)^{-L(0)} e^{-\sum_{k\in\mathbb{Z}_+}(a_0^{(-n+1)})^{-k}A_k^{(-1)}L(-k)} \otimes \cdots \otimes \\
&\left(\frac{a_0^{(-n+2)}}{a_0^{(-n+1)}}\right)^{-L(0)} e^{-\sum_{k\in\mathbb{Z}_+}(a_0^{(-n+1)})^{-k}A_k^{(-n+2)}L(-k)} \otimes \left(\frac{1}{a_0^{(-n+1)}}\right)^{-L(0)} \\
&e^{-\sum_{k\in\mathbb{Z}_+}(a_0^{(-n+1)})^{-k}A_k^{(-n)}L(-k)} \otimes e^{-\sum_{k\in\mathbb{Z}_+}(a_0^{(-n+1)})^{-k}A_n^{(-n+1)}L(-k)}\Bigg) \\
&(\underbrace{Id_V\otimes\cdots\otimes Id_V}_{n-2}\otimes\Lambda(x_{n-1}))(\underbrace{Id_V\otimes\cdots\otimes Id_V}_{n-3}\otimes\Lambda(x_{n-2}))\cdots \\
&(Id_V\otimes\Lambda(x_2))\Lambda(x_1)e^{a_0^{(-n+1)}x_{n-1}L(1)}e^{-\sum_{k\in\mathbb{Z}_+}(a_0^{(-n+1)})^{k}A_k^{(1)}L(k)} \\
&(a_0^{(1)}a_0^{(-n+1)})^{-L(0)}v\rangle\Big|_{x_{n-1}=-a_0^{(-n+1)}z_{-n+1},x_{n-2}=a_0^{(-n+1)}(z_{-n+2}-z_{-n+1}),\ldots,x_1=a_0^{(-n+1)}(z_{-1}-z_{-n+1})} \\
&= \langle v', \sigma(\mu_n^{\Lambda}(Q))v\rangle.
\end{aligned}
$$

where the last equality comes from recalling (4.3.2).

Thus for all $\sigma\in S_n$ and $Q\in K^*(n)$

$$\sigma(\mu_n(Q)) = \mu_n(\sigma(Q))$$

completing the proof of axiom 4.

5. Sewing: To prove the sewing axiom, consider $Q_1\in K^*(m)$ and $Q_2\in K^*(n)$ and $1\le i\le n$. Our proof will follow the technique of [B] and consider 8 steps. Step (a) will be the case that $m=0,1,\ n=1$, (b) - (f) will cover special cases needed for the induction, (g) is induction on m, and (h) is induction on n.

Throughout these steps, we will make use of the basis $\{e_{l^{(k)}}^{(k)}|l^{(k)}=1,\ldots,\dim V_{(k)};\ k\in\mathbb{Z}\}$ of V and $\{(e_{l^{(k)}}^{(k)})^*|l^{(k)}=1,\ldots,\dim V_{(k)};\ k\in\mathbb{Z}\}$ the corresponding dual basis.

Step (a): To prove the case $m=1,\ n=0,1,\ i=1$ consider

$$Q_1 = (A^{(-1)},(a_0^{(1)},A^{(1)}))\in K^*(1)=K(1),$$

$$Q_2 = (B^{(-1)},(b_0^{(1)},B^{(1)}))\in K^*(1)=K(1),$$

where $Q_1\ {}_1\infty_{-1}\ Q_2$ exists.

This case is proven in[H2]. Pages 121 to 123 there show that

$$\langle v',\mu_1^{\Lambda}(Q_1\ {}_1\infty_{-1}\ Q_2)v\rangle = \langle v',(\mu_1^{\Lambda}(Q_1)_1 *_{-1} \mu_1^{\Lambda}(Q_2)v\rangle e^{-\Gamma(A^{(1)},B^{(-1)},a_0^{(1)})C}$$

for all $v' \in V'$ and $v \in V$. The case $m = 0$, $n = 1$ is similar, but without needing $v' \in V'$.

Step (b): We now consider the case $i = 2$ with

$$Q_1 = (A^{(-1)}, (a_0^{(1)}, A^{(1)})) \in K^*(1),$$

$$Q_2 = (\zeta^{-1}; \mathbf{0}, (b_0^{(-1)}, B^{(-1)}), (b_0^{(1)}, B^{(1)})) \in K^*(2),$$

defined such that $Q_1\ {}_1\infty_{-i}\ Q_2$ exists.

$$\begin{aligned}
&\langle v', (\mu_1^{\Lambda}(Q_1)_1 *_{-i} \mu_2^{\Lambda}(Q_2))_t v\rangle \\
&= \sum_{k\in\mathbb{Z}} \sum_{l^{(k)}=1}^{\dim V_{(k)}} \langle v', \left(Id_V \otimes \left(e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-1)} L(-j)} e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)} (a_0^{(1)})^{-L(0)}\right) e_{l^{(k)}}^{(k)} (e_{l^{(k)}}^{(k)})^*\right) \\
&\quad \left((b_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-1)} L(-j)} \otimes Id_V\right) \Lambda(x) e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(1)} L(j)} (b_0^{(1)})^{-L(0)} v\rangle\Big|_{x=\zeta} t^k \\
&= \sum_{k\in\mathbb{Z}} \sum_{l^{(k)}=1}^{\dim V_{(k)}} \langle v', \left(Id_V \otimes \left(e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-1)} L(-j)} e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)} (\frac{t}{a_0^{(1)}})^{L(0)}\right) e_{l^{(k)}}^{(k)} (e_{l^{(k)}}^{(k)})^*\right) \\
&\quad \left((b_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-1)} L(-j)} \otimes Id_V\right) \Lambda(x)\, e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(1)} L(j)} (b_0^{(1)})^{-L(0)} v\rangle\Big|_{x=\zeta} \\
&= \langle v', \left(Id_V \otimes e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-1)} L(-j)} e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)} (a_0^{(1)} t^{-1})^{-L(0)}\right) \\
&\quad \left((b_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-1)} L(-j)} \otimes Id_V\right) \Lambda(x)\, e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(1)} L(j)} (b_0^{(1)})^{-L(0)} v\rangle\Big|_{x=\zeta} \\
&= \langle v', \left((b_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-1)} L(-j)} \otimes e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-1)} L(-j)}\right) \\
&\quad \left(Id_V \otimes e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)} (a_0^{(1)} t^{-1})^{-L(0)}\right) \Lambda(x) (a_0^{(1)} t^{-1})^{L(0)} e^{\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)} \\
&\quad e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)} (a_0^{(1)} t^{-1})^{-L(0)}\, e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(1)} L(j)} (b_0^{(1)})^{-L(0)} v\rangle\Big|_{x=\zeta}.
\end{aligned}$$

On the other hand, if we consider Q_2 as above and

$$Q_1(t) = (A^{(-1)}, (a_0^{(1)} t^{-1}, A^{(1)})) \in K^*(1),$$

then by Example 4.5.11 we observe that the image under μ_2^{Λ} of the sewing

$Q_1(t)\ {}_1\infty_{-2}\ Q_2$ exists for $t = 1$ and is given by

$$\begin{aligned}
&\mu_2^{\Lambda}(Q_1(t)\ {}_1\infty_{-2}\ Q_2)\\
&\quad = \langle v', \Big((b_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-1)} L(-j)} (a_0^{(1)} t^{-1})^{L(0)} e^{-\Theta_0^{(2)} L(0)}\\
&\quad\quad e^{-\sum_{j\in\mathbb{Z}_+} (a_0^{(1)} t^{-1})^{-j} \Theta_j^{(2)} L(-j)} \otimes e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-1)} L(-j)} \Big)\\
&\quad\quad \Lambda(a_0^{(1)} t^{-1} \hat{f}_2(x)) e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)} (a_0^{(1)} t^{-1})^{-L(0)}\ e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(1)} L(j)} (b_0^{(1)})^{-L(0)} v \rangle \Big|_{x=\zeta},
\end{aligned}$$

where $\Theta_j^{(2)}$, for $j \in \mathbb{N}$, and $\hat{f}_2(x)$ are considered as functions of $a_0^{(1)}$, $A^{(1)}$ and x. It therefore remains to show that

$$\begin{aligned}
&\Big(Id_V \otimes e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)} (a_0^{(1)} t^{-1})^{-L(0)} \Big) \Lambda(x) (a_0^{(1)} t^{-1})^{L(0)} e^{\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)}\\
&\quad = \Big((a_0^{(1)} t^{-1})^{L(0)} e^{-\Theta_0^{(2)} L(0)} e^{-\sum_{j\in\mathbb{Z}_+} (a_0^{(1)} t^{-1})^{-j} \Theta_j^{(2)} L(-j)} \otimes Id_V \Big) \Lambda(a_0^{(1)} t^{-1} \hat{f}_2(x)).
\end{aligned} \tag{8.2.3}$$

Using (7.2.1), (7.2.15), and Proposition 5.3.2, we observe that the right-hand side of (8.2.3) is equal to

$$\begin{aligned}
&\Big((a_0^{(1)} t^{-1})^{L(0)} e^{-\Theta_0^{(2)} L(0)} e^{-\sum_{j\in\mathbb{Z}_+} (a_0^{(1)} t^{-1})^{-j} \Theta_j^{(2)} L(-j)} e^{a_0^{(1)} t^{-1} (-x + \hat{f}_2(x)) L(1)} \otimes Id_V \Big)\\
&\quad \Lambda(a_0^{(1)} t^{-1} x)\\
&= \Big(e^{-\Theta_0^{(2)} L(0)} e^{-\sum_{j\in\mathbb{Z}_+} \Theta_j^{(2)} L(-j)} e^{(-x+\hat{f}_2(x)) L(1)} (a_0^{(1)} t^{-1})^{L(0)} \otimes Id_V \Big) \Lambda(a_0^{(1)} t^{-1} x)\\
&= \Big(e^{\sum_{k=-1}^{\infty} \left(\sum_{j\in\mathbb{Z}_+} (a_0^{(1)} t^{-1})^{-j} A_j^{(1)} \binom{-j+1}{k+1} x^{-j-k} \right) L(-k)} (a_0^{(1)} t^{-1})^{L(0)} \otimes Id_V \Big) \Lambda(a_0^{(1)} t^{-1} x)\\
&= \Big((a_0^{(1)} t^{-1})^{L(0)} \otimes Id_V \Big) \Big(e^{\sum_{k=-1}^{\infty} \left(\sum_{j\in\mathbb{Z}_+} (a_0^{(1)} t^{-1})^{-j-k} A_j^{(1)} \binom{-j+1}{k+1} x^{-j-k} \right) L(-k)} \otimes Id_V \Big)\\
&\quad \Lambda(a_0^{(1)} t^{-1} x).
\end{aligned}$$

On the other hand, using (7.2.11), we see that the left-hand side of (8.2.3) is equal to

$$\begin{aligned}
&\Big(Id_V \otimes e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)} \Big) \Big((a_0^{(1)} t^{-1})^{L(0)} \otimes Id_V \Big) \Lambda(a_0^{(1)} t^{-1} x) e^{\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)}\\
&= \Big((a_0^{(1)} t^{-1})^{L(0)} \otimes Id_V \Big) \Big(Id_V \otimes e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)} \Big) \Lambda(a_0^{(1)} t^{-1} x) e^{\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)}.
\end{aligned}$$

Thus it suffices to show that

$$\left(\sum_{k=-1}^{\infty}\left(\sum_{j\in\mathbb{Z}_+}(a_0^{(1)}t^{-1})^{-j-k}A_j^{(1)}\binom{-j+1}{k+1}x^{-j-k}\right)L(-k)\otimes Id_V\right)\mathcal{Y}(a_0^{(1)}t^{-1}x)$$

$$=\mathcal{Y}(a_0^{(1)}t^{-1}x)\sum_{j\in\mathbb{Z}_+}A_j^{(1)}L(j)-\left(Id_V\otimes\sum_{j\in\mathbb{Z}_+}A_j^{(1)}L(j)\right)\mathcal{Y}(a_0^{(1)}t^{-1}x). \quad (8.2.4)$$

Define

$$h(x_1)=\sum_{j\in\mathbb{Z}_+}A_j^{(1)}x_1^{-j+1}$$

and notice that

$$\mathrm{Res}_{x_1}(h(x_1)\sum_{m\in\mathbb{Z}}L(m)x_1^{m-2})=\sum_{j\in\mathbb{Z}_+}A_j^{(1)}L(j).$$

Now recalling (7.2.2) and observing that $h(x_1)$ commutes with all other terms, we observe that the right-hand side of (8.2.4) is

$$\mathrm{Res}_{x_1}\Bigg(\mathcal{Y}(a_0^{(1)}t^{-1}x)h(x_1)\sum_{m\in\mathbb{Z}}L(m)x_1^{m-2}$$
$$-\left(Id_V\otimes h(x_1)\sum_{m\in\mathbb{Z}}L(m)x_1^{m-2}\right)\mathcal{Y}(a_0^{(1)}t^{-1}x)\Bigg)$$
$$=\mathrm{Res}_{x_1}\mathrm{Res}_{x_0}(a_0^{(1)}t^{-1}x)^{-1}\delta\left(\frac{x_1-x_0}{a_0^{(1)}t^{-1}x}\right)\left(h(x_1)\sum_{m\in\mathbb{Z}}L(m)x_0^{m-2}\otimes Id_V\right)$$
$$\mathcal{Y}(a_0^{(1)}t^{-1}x)$$
$$=\mathrm{Res}_{x_1}\mathrm{Res}_{x_0}x_1^{-1}\delta\left(\frac{a_0^{(1)}t^{-1}x+x_0}{x_1}\right)\left(\sum_{j\in\mathbb{Z}_+}A_j^{(1)}x_1^{-j+1}\sum_{m\in\mathbb{Z}}L(m)x_0^{m-2}\otimes Id_V\right)$$
$$\mathcal{Y}(a_0^{(1)}t^{-1}x)$$
$$=\mathrm{Res}_{x_0}\left(\sum_{j\in\mathbb{Z}_+}A_j^{(1)}(a_0^{(1)}t^{-1}x+x)^{-j+1}\sum_{m\in\mathbb{Z}}L(m)x_0^{m-2}\otimes Id_V\right)\mathcal{Y}(a_0^{(1)}t^{-1}x)$$
$$=\mathrm{Res}_{x_0}\left(\sum_{j\in\mathbb{Z}_+}A_j^{(1)}\sum_{k=-1}^{\infty}\binom{-j+1}{k+1}(a_0^{(1)}t^{-1}x)^{-j-k}x_0^{k+1}\sum_{m\in\mathbb{Z}}L(m)x_0^{m-2}\otimes Id_V\right)$$
$$\mathcal{Y}(a_0^{(1)}t^{-1}x).$$

But this is equal to the left-hand side of (8.2.4). Therefore, since $\mu_2^{\Lambda}(Q_1(t)\ {}_1\infty_{-2}\ Q_2)$ exists when $t=1$,

$$\begin{aligned}\mu_2^{\Lambda}(Q_1(1)\ {}_1\infty_{-2}\ Q_2) &= \langle v', (\mu_1^{\Lambda}(Q_1)\ {}_1*_{-2}\ \ \mu_2^{\Lambda}(Q_2))_t\ v\rangle|_{t=1} \\ &= \langle v', (\mu_1^{\Lambda}(Q_1)\ {}_1*_{-2}\ \ \mu_2^{\Lambda}(Q_2))v\rangle.\end{aligned}$$

Step (c): Let $i=2$,

$$Q_1 = (A^{(-1)}, (a_0^{(1)}, A^{(1)})) \in K^*(1),$$

$$Q_2 = (\zeta^{-1}; B^{(-2)}, (b_0^{(-1)}, B^{(-1)}), (b_0^{(1)}, B^{(1)})) \in K^*(2),$$

where $Q_1\ {}_1\infty_{-i}\ Q_2$ exists. Using steps (a) and (b), Propositions 4.3.1 and 5.2.1, and Lemma 8.2.1

$$\begin{aligned}
&\mu_2^{\Lambda}(Q_1\ {}_1\infty_{-2}\ Q_2) \\
&= \mu_2^{\Lambda}((Q_1)\ {}_1\infty_{-2}\ ((B^{(-2)}, (1,\mathbf{0}))\ {}_1\infty_{-2}(\zeta^{-1}; \mathbf{0}, (b_0^{(-1)}, B^{(-1)}), (b_0^{(1)}, B^{(1)})))) \\
&= \mu_2^{\Lambda}(((Q_1)\ {}_1\infty_{-1}\ (B^{(-2)}, (1,\mathbf{0})))\ {}_1\infty_{-2}(\zeta^{-1}; \mathbf{0}, (b_0^{(-1)}, B^{(-1)}), (b_0^{(1)}, B^{(1)}))) \\
&= \mu_1^{\Lambda}((Q_1)\ {}_1\infty_{-1}\ (B^{(-2)}, (1,\mathbf{0})))\ {}_1*_{-2}\ \mu_2^{\Lambda}((\zeta^{-1}; \mathbf{0}, (b_0^{(-1)}, B^{(-1)}), (b_0^{(1)}, B^{(1)}))) \\
&= ((\mu_1^{\Lambda}(Q_1)\ {}_1*_{-1}\ \ \mu_1^{\Lambda}((B^{(-2)}, (1,\mathbf{0}))))\ {}_1*_{-2}\ \mu_2^{\Lambda}((\zeta^{-1}; \mathbf{0}, (b_0^{(-1)}, B^{(-1)}), (b_0^{(1)}, B^{(1)})))) \\
&\qquad e^{-\Gamma(A^{(1)}, B^{(-2)}, a_0^{(1)})C} \\
&= (\mu_1^{\Lambda}(Q_1)\ {}_1*_{-2}\ \ (\mu_1^{\Lambda}((B^{(-2)}, (1,\mathbf{0})))\ {}_1*_{-2}\ \mu_2^{\Lambda}((\zeta^{-1}; \mathbf{0}, (b_0^{(-1)}, B^{(-1)}), (b_0^{(1)}, B^{(1)}))))) \\
&\qquad e^{-\Gamma(A^{(1)}, B^{(-2)}, a_0^{(1)})C} \\
&= (\mu_1^{\Lambda}(Q_1)\ {}_1*_{-2}\ \ \mu_2^{\Lambda}((B^{(-2)}, (1,\mathbf{0}))\ {}_1\infty_{-2}\ (\zeta^{-1}; \mathbf{0}, (b_0^{(-1)}, B^{(-1)}), (b_0^{(1)}, B^{(1)})))) \\
&\qquad e^{-\Gamma(A^{(1)}, B^{(-i)}, a_0^{(1)})C} \\
&= (\mu_1^{\Lambda}(Q_1)\ {}_1*_{-2}\ \ \mu_2^{\Lambda}(Q_2))e^{-\Gamma(A^{(1)}, B^{(-2)}, a_0^{(1)})C}
\end{aligned}$$

Step (d): Let $i=1$,

$$Q_1 = (A^{(-1)}, (a_0^{(1)}, A^{(1)})) \in K^*(1),$$

$$Q_2 = (\zeta^{-1}; B^{(-2)}, (b_0^{(-1)}, B^{(-1)}), (b_0^{(1)}, B^{(1)})) \in K^*(2),$$

where $Q_1 \ {}_1\infty_{-i} \ Q_2$ exists.

Let $\sigma \in S_2$ be the transposition of two elements. We use the naturality of permutations (Propositions 4.3.2 and 5.2.2 and axiom 4 for the GVOC), along with part (c) to observe the following:

$$\begin{aligned}\mu_2^{\Lambda}(Q_1 \ {}_1\infty_{-1} \ Q_2) &= \mu_2^{\Lambda}(\sigma(Q_1 \ {}_1\infty_{-2} \ \sigma \ Q_2)) \\ &= \sigma(\mu_1^{\Lambda}(Q_1) \ {}_1*_{-2} \ \mu_2^{\Lambda}(\sigma \ Q_2))e^{-\Gamma(A^{(1)},B^{(-1)}((b_0^{(-1)})^{-1}),a_0^{(1)})C} \\ &= \sigma(\mu_1^{\Lambda}(Q_1) \ {}_1*_{-2} \ \sigma\mu_2^{\Lambda}(Q_2))e^{-\Gamma(A^{(1)},B^{(-1)}((b_0^{(-1)})^{-1}),a_0^{(1)})C} \\ &= (\mu_1^{\Lambda}(Q_1) \ {}_1*_{-1} \ \mu_2^{\Lambda}(Q_2))e^{-\Gamma(A^{(1)},B^{(-1)}((b_0^{(-1)})^{-1}),a_0^{(1)})C}.\end{aligned}$$

The careful reader will have noticed the inputs for Γ are modified to $A^{(1)}$, $B^{(-1)}((b_0^{(-1)})^{-1})$, $a_0^{(1)}$ because we are working with $\sigma \ Q_2$ instead of Q_2 as is made explicit in (4.3.2). Via Proposition 4.2.1 in [H2], we have

$$\Gamma(A^{(1)}, B^{(-1)}((b_0^{(-1)})^{-1}), a_0^{(1)}) = \Gamma(A^{(1)}, B^{(-1)}, a_0^{(1)}b_0^{(-1)}).$$

Step (e): Let $i = n$,

$$Q_1 = (z^{-1}; A^{(-2)}, (a_0^{(-1)}, A^{(-1)}), (1, \mathbf{0})) \in K^*(2),$$

$$\begin{aligned}Q_2 = (\zeta_{-n+1}^{-1}, \ldots, \zeta_{-1}^{-1}; B^{(-n)}, (b_0^{(-n+1)}, B^{(-n+1)}), \ldots, \\ (b_0^{(-1)}, B^{(-1)}), (b_0^{(1)}, B^{(1)})) \in K^*(n),\end{aligned}$$

where $Q_1\ {}_1\infty_{-n}\ Q_2$ exists. Then

$$
\begin{aligned}
&\langle v', (\mu_2^{\Lambda}(Q_1)_1 *_{-n} \mu_n^{\Lambda}(Q_2))_t v\rangle\\
&= \sum_{k\in\mathbb{Z}} \sum_{l^{(k)}=1}^{\dim V_{(k)}} \langle v', \Bigg(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-1} \otimes \Bigg(\Bigg(\Big((a_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-1)} L(-j)}\\
&\quad \otimes e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-2)} L(-j)}\Big) \Lambda(x) t^{L(0)} e^{(k)}_{l^{(k)}} (e^{(k)}_{l^{(k)}})^*\Bigg)\Bigg|_{x=z}\Bigg)\\
&\quad \Bigg((b_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-1)} L(-j)} \otimes \cdots \otimes (b_0^{(-n+1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-n+1)} L(-j)}\\
&\quad \otimes e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-n)} L(-j)}\Bigg) (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes \Lambda(y_{n-1}))\\
&\quad \cdots (Id_V \otimes \Lambda(y_2))\Lambda(y_1)\ e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(1)} L(j)} (b_0^{(1)})^{-L(0)} v\rangle\Big|_{y_i=\zeta_{-i}}\\
&= \sum_{k\in\mathbb{Z}} \sum_{l^{(k)}=1}^{\dim V_{(k)}} \langle v', \Bigg((b_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-1)} L(-j)} \otimes \cdots \otimes (b_0^{(-n+1)})^{-L(0)}\\
&\quad e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-n+1)} L(-j)} \otimes (a_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-1)} L(-j)} \otimes e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-2)} L(-j)}\Bigg)\\
&\quad (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-1} \otimes \Lambda(x) t^{L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-n)} L(-j)})\\
&\quad (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes \Lambda(y_{n-1}))\\
&\quad \cdots (Id_V \otimes \Lambda(y_2))\Lambda(y_1)\ e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(1)} L(j)} (b_0^{(1)})^{-L(0)} v\rangle\Big|_{y_i=\zeta_{-i},\ x=z}\\
&= \sum_{k\in\mathbb{Z}} \sum_{l^{(k)}=1}^{\dim V_{(k)}} \langle v', \Bigg((b_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-1)} L(-j)} \otimes \cdots \otimes (b_0^{(-n+1)})^{-L(0)}\\
&\quad e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-n+1)} L(-j)} \otimes (a_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-1)} L(-j)} \otimes e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-2)} L(-j)}\\
&\quad t^{L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-n)} L(-j)})\Bigg) (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-1} \otimes (Id_V \otimes e^{\sum_{j\in\mathbb{Z}_+} B_j^{(-n)} L(-j)} t^{-L(0)})\\
&\quad \Lambda(x) t^{L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-n)} L(-j)})(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes \Lambda(y_{n-1}))\\
&\quad \cdots (Id_V \otimes \Lambda(y_2))\Lambda(y_1)\ e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(1)} L(j)} (b_0^{(1)})^{-L(0)} v\rangle\Big|_{y_i=\zeta_{-i},\ x=z}.
\end{aligned}
$$

On the other hand, recalling Example 4.5.8, we have $\Theta_j^{(1)}(B^{(-n)}, x)$, for $j \in \mathbb{N}$, and $\hat{f}_1(x)$ which are functions of $B^{(-n)}$ and x, and satisfy

$$\begin{aligned}
&\langle v', \mu_{n+1}^{\Lambda}(Q_1 \; {}_1\infty_{-n} \; Q_2)v\rangle \\
&= \langle v', \Big((b_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-1)} L(-j)} \otimes \cdots \otimes (b_0^{(-n+1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-n+1)} L(-j)} \\
&\otimes (a_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-1)} L(-j)} e^{-\Theta_0^{(1)}(B^{(-n)},x)L(0)} e^{-\sum_{j\in\mathbb{Z}_+} \Theta_j^{(1)}(B^{(-n)},x)L(-j)} \otimes \\
&e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-2)} L(-j)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-n)} L(-j)} \Big) (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-1} \otimes \Lambda(x_n)) \cdots \\
&(Id_V \otimes \Lambda(x_2)) \; \Lambda(x_1) e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(1)} L(j)} (b_0^{(1)})^{-L(0)} v\rangle \Big|_{x_1=\zeta_{-1},\ldots,\; x_{-n+1}=\zeta_{-n+1},\; x_{-n}=\hat{f}_1^{-1}(z)} \\
&= \langle v', \Big((b_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-1)} L(-j)} \otimes \cdots \otimes (b_0^{(-n+1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-n+1)} L(-j)} \\
&\otimes (a_0^{(-1)})^{-L(0)} e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-1)} L(-j)} \otimes e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-2)} L(-j)} t^{L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-n)} L(-j)} \Big) \\
&(\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-1} \otimes (e^{-\Theta_0^{(1)}(B^{(-n)}(t),x)L(0)} e^{-\sum_{j\in\mathbb{Z}_+} \Theta_j^{(1)}(B^{(-n)}(t),x)L(-j)} t^{L(0)} \otimes Id_V) \\
&\Lambda(t\hat{f}_1^{-1}(x)) \cdots (Id_V \otimes \Lambda(x_2)) \Lambda(x_1) e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(1)} L(j)} \\
&(b_0^{(1)})^{-L(0)} v \; \rangle|_{t=1} \Big|_{x_1=\zeta_{-1},\ldots,\; x_{-n+1}=\zeta_{-n+1},\; x_{-n}=z_{-1}} .
\end{aligned}$$

Thus it is enough to show that

$$\begin{aligned}
&\left(Id_V \otimes e^{\sum_{j\in\mathbb{Z}_+} B_j^{(-n)} L(-j)} t^{-L(0)} \right) \Lambda(x) t^{L(0)} e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-n)} L(-j)} \\
&\quad = (e^{-\Theta_0^{(1)}(B^{(-n)}(t),x)L(0)} e^{-\sum_{j\in\mathbb{Z}_+} \Theta_j^{(1)}(B^{(-n)}(t),x)L(-j)} t^{L(0)} \otimes Id_V) \Lambda(t\hat{f}_1^{-1}(x)). \quad (8.2.5)
\end{aligned}$$

Using (7.2.1) and appealing to Proposition 5.3.1, we observe that the right-hand side of (8.2.5) is equal to

$$\begin{aligned}
&\left(e^{-\Theta_0^{(1)}(B^{(-n)}(t),x)L(0)} e^{-\sum_{j\in\mathbb{Z}_+} \Theta_j^{(1)}(B^{(-n)}(t),x)L(-j)} t^{L(0)} \; e^{-t(-x+\hat{f}_1^{-1}(x))L(1)} \otimes Id_V \right) \Lambda(tx) \\
&= \left(e^{-\Theta_0^{(1)}(B^{(-n)}(t),x)L(0)} e^{-\sum_{j\in\mathbb{Z}_+} \Theta_j^{(1)}(B^{(-n)}(t),x)L(-j)} \; e^{-(-x+\hat{f}_1^{-1}(x))L(1)} t^{L(0)} \otimes Id_V \right) \Lambda(tx) \\
&= \left(e^{-\sum_{k=-1}^{\infty} \left(\sum_{j\in\mathbb{Z}_+} \binom{j+1}{k+1} t^j B_j^{(-1)} x^{j-k} \right) L(-k)} t^{L(0)} \otimes Id_V \right) \Lambda(tx) \\
&= \left(t^{L(0)} e^{-\sum_{k=-1}^{\infty} \left(\sum_{j\in\mathbb{Z}_+} \binom{j+1}{k+1} t^{j-k} B_j^{(-1)} x^{j-k} \right) L(-k)} \otimes Id_V \right) \Lambda(tx).
\end{aligned}$$

On the other hand, using (7.2.11), we see that the left-hand side of (8.2.5) is equal to

$$\left(t^{L(0)} \otimes e^{\sum_{j\in\mathbb{Z}_+} B_j^{(-n)} L(-j)}\right) \Lambda(tx) e^{-\sum_{j\in\mathbb{Z}_+} B_j^{(-n)} L(-j)}.$$

Thus it suffices to show that

$$\begin{aligned}
&\left(-\sum_{k=-1}^{\infty}\left(\sum_{j\in\mathbb{Z}_+}\binom{j+1}{k+1} t^{j-k} B_j^{(-1)} x^{j-k}\right) L(-k)\otimes Id_V\right)\Lambda(tx)\\
&\qquad = \left(Id_V \otimes \sum_{j\in\mathbb{Z}_+} B_j^{(-n)} L(-j)\right)\Lambda(tx) - \Lambda(tx)\sum_{j\in\mathbb{Z}_+} B_j^{(-n)} L(-j). \quad (8.2.6)
\end{aligned}$$

Define

$$g(x_1) = \sum_{j\in\mathbb{Z}_+} B_j^{(-n)} x_1^{j+1}$$

and notice that

$$\operatorname{Res}_{x_1}(g(x_1)\sum_{m\in\mathbb{Z}} L(m) x_1^{m-2}) = \sum_{j\in\mathbb{Z}_+} B_j^{(-n)} L(-j).$$

Now recalling (7.2.2) and observing that $g(x_1)$ commutes with all other terms, we calculate that the right-hand side of (8.2.6) is

$$\begin{aligned}
&\operatorname{Res}_{x_1}\left(\left(Id_V \otimes g(x_1)\sum_{m\in\mathbb{Z}} L(m)x_1^{m-2}\right)\Lambda(tx) - \Lambda(tx) g(x_1)\sum_{m\in\mathbb{Z}} L(m) x_1^{m-2}\right)\\
&= -\operatorname{Res}_{x_1}\operatorname{Res}_{x_0}(tx)^{-1}\delta\left(\frac{x_1-x_0}{tx}\right)\left(g(x_1)\sum_{m\in\mathbb{Z}} L(m)x_0^{m-2}\otimes Id_V\right)\Lambda(tx)\\
&= -\operatorname{Res}_{x_1}\operatorname{Res}_{x_0} x_1^{-1}\delta\left(\frac{tx+x_0}{x_1}\right)\left(\sum_{j\in\mathbb{Z}_+} B_j^{(-n)} x_1^{j+1}\sum_{m\in\mathbb{Z}} L(m)x_0^{m-2}\otimes Id_V\right)\Lambda(tx)\\
&= -\operatorname{Res}_{x_0}\left(\sum_{j\in\mathbb{Z}_+} B_j^{(-n)}\sum_{k=-1}^{\infty}\binom{j+1}{k+1}(tx)^{j-k}x_0^{k+1}\sum_{m\in\mathbb{Z}} L(m)x_0^{m-2}\otimes Id_V\right)\Lambda(tx).
\end{aligned}$$

But this is equal to the left-hand side of (8.2.6). Therefore, since $\mu_{n+1}^{\Lambda}(Q_1\ {}_1\infty_{-n}\ Q_2)$ exists,

$$\begin{aligned}
\mu_{n+1}^{\Lambda}(Q_1\ {}_1\infty_{-n}\ Q_2) &= \langle v', (\mu_2^{\Lambda}(Q_1)\ {}_1*_{-n}\ \mu_n^{\Lambda}(Q_2))_t\, v\rangle|_{t=1}\\
&= \langle v', (\mu_2^{\Lambda}(Q_1)\ {}_1*_{-n}\ \mu_n^{\Lambda}(Q_2))v\rangle.
\end{aligned}$$

Step (f): Let $1 \le i \le n$,

$$Q_1 = (z_{-1}^{-1}; A^{(-2)}, (a_0^{(-1)}, A^{(-1)}), (1, \mathbf{0})) \in K^*(2),$$

$$Q_2 = (\zeta_{-n+1}^{-1}, \ldots, \zeta_{-1}^{-1}; B^{(-n)}, (b_0^{(-n+1)}, B^{(-n+1)}), \ldots, \\ (b_0^{(-1)}, B^{(-1)})), (b_0^{(1)}, B^{(1)})) \in K^*(n),$$

such that $Q_1\ {}_1\infty_{-i}\ Q_2$ exists. If σ is defined to be the transposition $(i\ n)$, then following the proof of Step (d), with τ the permutation $(i\ n+1\ i+1\ i+2 \ldots n)$,

$$\begin{aligned} \mu_{n+1}^{\Lambda}(Q_1\ {}_1\infty_{-i}\ Q_2) &= \mu_{n+1}^{\Lambda}(\tau(Q_1\ {}_1\infty_{-n}\ \sigma Q_2)) \\ &= \tau(\mu_{n+1}^{\Lambda}(Q_1\ {}_1\infty_{-n}\ \sigma Q_2)) \\ &= \tau(\mu_2^{\Lambda}(Q_1)\ {}_1*_{-n}\ \mu_n^{\Lambda}(\sigma\ Q_2)) \\ &= \tau(\mu_2^{\Lambda}(Q_1)\ {}_1*_{-n}\ \sigma\mu_n^{\Lambda}(Q_2)) \\ &= (\mu_2^{\Lambda}(Q_1)\ {}_1*_{-i}\ \mu_n^{\Lambda}(Q_2)). \end{aligned}$$

where Step (e) gives the key equality.

Step (g): We will now use induction on m for $m \ge 2$. Assume that for all $\ell < m$ the sewing axiom holds and let $1 \le i \le n$,

$$Q_1 = (z_{-m+1}^{-1}, \ldots, z_{-1}^{-1}; A^{(-m)}, (a_0^{(-m+1)}, A^{(-m+1)}), \ldots, \\ (a_0^{(-1)}, A^{(-1)})), (a_0^{(1)}, A^{(1)})) \in K^*(m),$$

$$Q_2 = (\zeta_{-n+1}^{-1}, \ldots, \zeta_{-1}^{-1}; B^{(-n)}, (b_0^{(-n+1)}, B^{(-n+1)}), \ldots, \\ (b_0^{(-1)}, B^{(-1)})), (b_0^{(1)}, B^{(1)})) \in K^*(n),$$

such that $Q_1\ {}_1\infty_{-i}\ Q_2$ exists. Since both sides of (6.1.1) are analytic in z_{-1}, …, z_{-m+1}, we need only consider the case

$$|z_{-1}|, \ldots, |z_{-m+2}| < |z_{-m+1}|.$$

We begin by decomposing Q_1 into

$$Q_1^+ = (z_{-m+1}^{-1}; A^{(-m)}, (a_0^{(-m+1)}, A^{(-m+1)}), (1, \mathbf{0})),$$

$$Q_1^- \quad = \quad (z_{-m+2}^{-1}, \ldots, z_{-1}^{-1}; \mathbf{0}, (a_0^{(-m+2)}, A^{(-m+2)}), \ldots, (a_0^{(-1)}, A^{(-1)}), (a_0^{(1)}, A^{(1)})).$$

Then, as in Step (c) (using associativity, the t-contraction and Lemma 8.2.1)

$$\begin{aligned}
&\mu_{m+n-1}^{\Lambda}(Q_1 \ {}_1\infty_{-i} Q_2) \\
&= \mu_{m+n-1}^{\Lambda}((Q_1^+ \ {}_1\infty_{-m+1} \ Q_1^-) \ {}_1\infty_{-i} Q_2) \\
&= \mu_{m+n-1}^{\Lambda}(Q_1^+ \ {}_1\infty_{-i-m+2} \ (Q_1^- \ {}_1\infty_{-i} Q_2)) \\
&= \mu_2^{\Lambda}(Q_1^+) \ {}_1*_{-i-m+2} \ \mu_{m+n-2}^{\Lambda}(Q_1^- \ {}_1\infty_{-i} Q_2) \\
&= (\mu_2^{\Lambda}(Q_1^+) \ {}_1*_{-i-m+2} \ (\mu_{m-1}^{\Lambda}(Q_1^-) \ {}_1*_{-i} \ \mu_n^{\Lambda}(Q_2)))e^{-\Gamma(A^{(1)}, B^{(-i)}, a_0^{(1)} b_0^{(-i)})C} \\
&= ((\mu_2^{\Lambda}(Q_1^+) \ {}_1*_{-m+1} \ \mu_{m-1}^{\Lambda}(Q_1^-)) \ {}_1*_{-i} \ \mu_n^{\Lambda}(Q_2))e^{-\Gamma(A^{(1)}, B^{(-i)}, a_0^{(1)} b_0^{(-i)})C} \\
&= (\mu_m^{\Lambda}(Q_1) \ {}_1*_{-i} \ \mu_n^{\Lambda}(Q_2))e^{-\Gamma(A^{(1)}, B^{(-i)}, a_0^{(1)} b_0^{(-i)})C}
\end{aligned}$$

where, for $m > 2$, the third, fourth and sixth equalities employ the inductive assumption. If $m = 2$, the third and sixth equalities follow from Step (f), while equality four uses induction.

Step (h): We will now use induction on n for $n \geq 2$. Assume that for all $\ell < n$ the sewing axiom holds and let $1 \leq i \leq n$,

$$\begin{aligned}
Q_1 = (z_{-m+1}^{-1}, \ldots, z_{-1}^{-1}; A^{(-m)}, (a_0^{(-m+1)}, A^{(-m+1)}), \ldots, \\
(a_0^{(-1)}, A^{(-1)}), (a_0^{(1)}, A^{(1)})) \in K^*(m),
\end{aligned}$$

$$\begin{aligned}
Q_2 = (\zeta_{-n+1}^{-1}, \ldots, \zeta_{-1}^{-1}; B^{(-n)}, (b_0^{(-n+1)}, B^{(-n+1)}), \ldots, \\
(b_0^{(-1)}, B^{(-1)}), (b_0^{(1)}, B^{(1)})) \in K^*(n),
\end{aligned}$$

such that $Q_1 \ {}_1\infty_{-i} \ Q_2$ exists. Following our Step (g) approach, we may consider Q_2 that decomposes into

$$Q_2^+ = (\zeta_{-n+1}^{-1}; B^{(-n)}, (b_0^{(-n+1)}, B^{(-n+1)}), (1, \mathbf{0})),$$

$$Q_2^- = (\zeta_{-1}^{-n+2}, \dots, \zeta_{-1}^{-1}; \mathbf{0}, (b_0^{(-n+2)}, B^{(-n+2)}), \dots, (b_0^{(-1)}, B^{(-1)})), (b_0^{(1)}, B^{(1)})).$$

There are three possibilities: (i) If $i = n$, then

$$\begin{aligned} Q_1 \ {}_1\infty_{-n} \ Q_2 &= Q_1 \ {}_1\infty_{-n} \ (Q_2^+ \ {}_1\infty_{-n+1} \ Q_2^-) \\ &= (Q_1 \ {}_1\infty_{-2} \ Q_2^+) \ {}_1\infty_{-n+1} \ Q_2^-. \end{aligned}$$

For $n > 2$, the sewing follows from the inductive assumption, associativity, the t-contraction and Lemma 8.2.1. For $n = 2$, we supplement induction with Steps (c) and (g).

(ii) If $i = n - 1$, then

$$\begin{aligned} Q_1 \ {}_1\infty_{-n+1} \ Q_2 &= Q_1 \ {}_1\infty_{-n+1} \ (Q_2^+ \ {}_1\infty_{-n+1} \ Q_2^-) \\ &= (Q_1 \ {}_1\infty_{-1} \ Q_2^+) \ {}_1\infty_{-n+1} \ Q_2^- \end{aligned}$$

For $n > 2$, the sewing follows from the inductive assumption, associativity, the t-contraction and Lemma 8.2.1. For $n = 2$, we supplement induction with Steps (d) and (g).

(iii) For $i < n - 1$,

$$\begin{aligned} Q_1 \ {}_1\infty_{-i} \ Q_2 &= Q_1 \ {}_1\infty_{-i} \ (Q_2^+ \ {}_1\infty_{-n+1} \ Q_2^-) \\ &= Q_2^+ \ {}_1\infty_{-m-n+2} \ (Q_1 \ {}_1\infty_{-i} \ Q_2^-). \end{aligned}$$

The sewing follows from the inductive assumption, Step (e), associativity, the t-contraction and Lemma 8.2.1. □

8.3 The categorical isomorphism between GVOCs and VOCs

Let C be a complex number, $\mathbf{V}^*(C)$ be the category of vertex operator coalgebras of rank C, and $\mathbf{G}^*(C)$ be the category of geometric vertex operator coalgebras of rank C. In the previous two sections we have defined functors

$$\begin{aligned} F_{\mathbf{V}^*(C)} : \mathbf{V}^*(C) &\to \mathbf{G}^*(C) \\ (V, \Lambda, c, \rho) &\mapsto (V, \mu^{\Lambda}) \end{aligned}$$

and

$$\begin{aligned} F_{\mathbf{G}^*(C)} : \mathbf{G}^*(C) \to& \mathbf{V}^*(C) \\ (V, \mu) \mapsto& (V, \mathit{\Lambda}_\mu, c_\mu, \rho_\mu). \end{aligned}$$

(We have shown explicitly that these two maps take objects to objects but since morphisms in both categories are simply linear maps and the operators in each object are linear, these functions must respect morphisms as well.) The main purpose of defining $F_{\mathbf{V}^*(C)}$ and $F_{\mathbf{G}^*(C)}$ is to show that they are inverses to each other and that the categories of VOCs and GVOCs are isomorphic.

Theorem 8.3.1. *The categories* $\mathbf{V}^*(C)$ *and* $\mathbf{G}^*(C)$ *are isomorphic. In particular, the functors* $F_{\mathbf{V}^*(C)}$ *and* $F_{\mathbf{G}^*(C)}$ *satisfy*

$$F_{\mathbf{G}^*(C)} F_{\mathbf{V}^*(C)} = 1_{\mathbf{V}^*(C)}, \tag{8.3.1}$$

$$F_{\mathbf{V}^*(C)} F_{\mathbf{G}^*(C)} = 1_{\mathbf{G}^*(C)} \tag{8.3.2}$$

where $1_{\mathbf{V}^*(C)}$ *and* $1_{\mathbf{G}^*(C)}$ *are the identity functors on* $\mathbf{V}^*(C)$ *and* $\mathbf{G}^*(C)$*, respectively.*

Proof. Since the second half of the theorem implies the first half, we will simply prove (8.3.1) and (8.3.2). First, observe that

$$c_\mu = \mu_0^{\mathit{\Lambda}}((1, \mathbf{0})) = c,$$

and

$$\rho_\mu = -\frac{d}{d\epsilon} \mu_0^{\mathit{\Lambda}}(1, (0, \epsilon, 0, 0, \ldots))\bigg|_{\epsilon=0} = -\frac{d}{d\epsilon} c e^{-\epsilon L(2)}\bigg|_{\epsilon=0} = cL(2) = \rho$$

by employing (7.2.13). For any $v' \in V \otimes V$, $v \in V$,

$$\begin{aligned} \mathrm{Res}_x x^n \langle v', \mathit{\Lambda}^\mu(x) v \rangle &= \mathrm{Res}_z z^n \langle v', \mu^{\mathit{\Lambda}}((z^{-1}; (1, \mathbf{0}), (1, \mathbf{0}), \mathbf{0}))(v) \rangle \\ &= \mathrm{Res}_z z^n \left(\langle v', \mathit{\Lambda}(x) v \rangle |_{x=z} \right) \\ &= \mathrm{Res}_x x^n \langle v', \mathit{\Lambda}(x) v \rangle. \end{aligned}$$

Thus (8.3.1) is verified.

As for (8.3.2), we need only verify that

$$\begin{aligned}
&\langle v', \mu_n(z_{-1}^{-1}, \dots, z_{-n+1}^{-1}; (a_0^{(1)}, A^{(1)}), (a_0^{(-1)}, A^{(-1)}), \dots, (a_0^{(-n+1)}, A^{(-n+1)}), A^{(-n)})v\rangle \\
&= \iota_{x\cdots n-1}^{-1} \langle v', \Big(e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-1)} L(-j)} (a_0^{(-1)})^{-L(0)} \otimes \cdots \otimes e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-n+1)} L(-j)} \\
&\quad (a_0^{(-n+1)})^{-L(0)} \otimes e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-n)} L(-j)} \Big) (\underbrace{Id_V \otimes \cdots \otimes Id_V}_{n-2} \otimes \varLambda_\mu(x_{n-1})) \cdots \\
&\quad (Id_V \otimes \varLambda_\mu(x_2)) \varLambda_\mu(x_1)\, e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)} (a_0^{(1)})^{-L(0)} v\rangle \Big|_{x_i = z_{-i}}. \qquad (8.3.3)
\end{aligned}$$

Since both sides of (8.3.3) define geometric vertex operator coalgebras, both satisfy the sewing axiom. Therefore, by Proposition 3.4.2, we need only prove (8.3.3) in the specialized cases

$$\langle v', \mu_2(z^{-1}; (1, \mathbf{0}), (1, \mathbf{0}), \mathbf{0})v\rangle = \langle v', \varLambda_\mu(x)v\rangle|_{x=z},$$

$$\mu_0((1, \mathbf{0}))v = c,$$

$$\langle v', \mu_1((a_0^{(1)}, A^{(1)}), A^{(-1)})v\rangle = \langle v', e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(-1)} L(-j)} e^{-\sum_{j\in\mathbb{Z}_+} A_j^{(1)} L(j)} (a_0^{(1)})^{-L(0)} v\rangle.$$

The first two of these equations are true by the definition of $F_{\mathbf{G}^*(C)}$. The third equation, the proof for $K^*(1)$, is identical to the proof for $K(1)$ in [H2] (Equation (5.4.31) in [H2]) since $K(1) = K^*(1)$ in the most specific sense, the definitions of $L(0)$ are the same and the meromorphicity axiom is the same in that special case.

□

CHAPTER 9

A FAMILY OF EXAMPLES OF VOCS

Now that we have a formal algebraic notion of a VOC and we have seen that this notion arises directly from the geometry of worldsheets, the next natural question to ask is: What do VOCs look like? While the properties in Sections 7.2 and 7.3 give some insight into that question, having examples is perhaps the most concrete way to investigate the VOC structure. In our final section, we will look at a family of examples arising from examples of VOAs with nondegenerate, invariant bilinear forms.

9.1 The definition of a VOA

Our examples are generated from examples of VOAs, so it is fitting to begin with a review of the definition of a VOA.

Definition 9.1.1. *A* vertex operator algebra (of rank C) *is a* $\mathbb{Z}$*-graded vector space (graded by* weights*)*

$$V = \coprod_{k\in\mathbb{Z}} V_{(k)}, \qquad \textit{for } v \in V_{(k)},\ k = wt\ v;$$

such that $\dim V_{(k)} < \infty$ *for* $k \in \mathbb{Z}$*, together a with linear map* $V \otimes V \to V[[x, x^{-1}]]$*, or equivalently,*

$$\begin{aligned} Y : V &\mapsto (End\ V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{k\in\mathbb{Z}} v_k x^{-k-1} \qquad (\textit{where } v_n \in End\ V), \end{aligned}$$

and equipped with two distinguished homogeneous vectors in V, $\mathbf{1}$ (the vacuum*) and ω (the* Virasoro element*), satisfying the following eight axioms:*

1. *Positive energy:*

$$V_{(k)} = 0 \qquad \text{for } k \text{ sufficiently small.} \tag{9.1.1}$$

2. *Left unit: For all $v \in V$*

$$Y(\mathbf{1}, x)v = v \tag{9.1.2}$$

3. *Creation: For all $v \in V$*

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \tag{9.1.3}$$

$$\lim_{x \to 0} Y(v, x)\mathbf{1} = v. \tag{9.1.4}$$

4. *Truncation: Given $v, w \in V$, then $v_k w = 0$ for k sufficiently large.*

5. *Jacobi identity: For all $v, w \in V$,*

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(v, x_1)Y(w, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(w, x_2)Y(v, x_1)$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(v, x_0)w, x_2). \tag{9.1.5}$$

6. *Virasoro Algebra: The Virasoro algebra bracket,*

$$[L(j), L(k)] = (j - k)L(j + k) + \frac{1}{12}(j^3 - j)\delta_{j,-k}C,$$

holds for $j, k \in \mathbb{Z}$, where

$$Y(\omega, x) = \sum_{k \in \mathbb{Z}} L(k)x^{-k-2}. \tag{9.1.6}$$

7. *Grading: For each $k \in \mathbb{Z}$ and $v \in V_{(k)}$*

$$L(0)v = kv. \tag{9.1.7}$$

8. *$L(-1)$-derivative: Given $v \in V$,*

$$\frac{d}{dx}Y(v, x) = Y(L(-1)v, x). \tag{9.1.8}$$

We denote a VOA either by V or by the quadruple $(V, Y, \mathbf{1}, \omega)$. A vector $v \in V_{(k)}$ for some $k \in \mathbb{Z}$ is said to be a *homogeneous vector of weight* k and we write wt $v = k$. A pair of basic properties of VOAs will be necessary for our discussion (cf. [FHL], [LL]):

$$Y(v,x)\mathbf{1} = e^{xL(-1)}v \qquad \text{for } v \in V,$$

$$\text{wt } v_k w = r + s - k - 1 \qquad \text{for } v \in V_{(r)},\ w \in V_{(s)}$$

9.2 A family of examples of VOCs

Let the vector space $V = \coprod_{k\in\mathbb{Z}} V_{(k)}$ be a module over the Virasoro algebra, $\mathcal{V} = \oplus_{j\in\mathbb{Z}}\mathbb{C}L(j) \oplus \mathbb{C}d$, such that for all homogeneous vectors $L(0)\ v = \text{wt }(v)v$. We will say that a bilinear form $(\cdot,\cdot)$ on V is *Virasoro preserving* if it satisfies the condition

$$(L(k)v_1, v_2) = (v_1, L(-k)v_2) \tag{9.2.1}$$

for all $k \in \mathbb{Z}$, $v_1, v_2 \in V$. In particular, for $k = 0$, Property (9.2.1) indicates that all Virasoro preserving bilinear forms are graded, i.e.

$$(V_{(k)}, V_{(\ell)}) = 0$$

for $k \neq \ell$. If V is a VOA, the bilinear form is said to be *invariant* if, for all $u, v, w \in V$,

$$(Y(u,x)v, w) = (u, Y(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w). \tag{9.2.2}$$

Any invariant bilinear form on V is Virasoro preserving ((2.31) in [L]).

Note that there is a natural extension of $(\cdot,\cdot) : V^{\otimes 2} \to \mathbb{C}$ to $(\cdot,\cdot) : V^{\otimes 4} \to \mathbb{C}$ given by $(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)(u_2, v_2)$, for $u_1, u_2, v_1, v_2 \in V$.

Theorem 9.2.1. *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra equipped with a non-degenerate and Virasoro preserving bilinear form $(\cdot, \cdot)$. Given the linear operators*

$$\begin{aligned} c : V &\to \mathbb{C} \\ v &\mapsto (v, \mathbf{1}), \end{aligned}$$

$$\begin{aligned} \rho : V &\to \mathbb{C} \\ v &\mapsto (v, \omega), \end{aligned}$$

and

$$\begin{aligned} \Lambda : V &\to (V \otimes V)[[x, x^{-1}]] \\ v &\mapsto \Lambda(x)v = \sum_{k \in \mathbb{Z}} \Delta_k(v) x^{-k-1}, \end{aligned}$$

defined by

$$(\Lambda(x)u, v \otimes w) = (u, Y(v, x)w), \tag{9.2.3}$$

the quadruple (V, Λ, c, ρ) is a vertex operator coalgebra.

Proof. We will show that all eight axioms for VOCs are satisfied.

1. Positive energy: Trivially satisfied.
2. Left counit: Given $u \in V$, for all $v \in V$

$$\begin{aligned} ((c \otimes Id_V)\Lambda(x)u, v) &= (\Lambda(x)u, \mathbf{1} \otimes v) \\ &= (u, Y(\mathbf{1}, x)v) \\ &= (u, v). \end{aligned}$$

Thus, by nondegeneracy, $(c \otimes Id_V)\Lambda(x)u = u$.

3. Cocreation: Given $u \in V$, for all $v \in V$

$$\begin{aligned} ((Id_V \otimes c)\Lambda(x)u, v) &= (\Lambda(x)u, v \otimes \mathbf{1}) \\ &= (u, Y(v, x)\mathbf{1}) \\ &= (u, e^{xL(-1)}v) \in \mathbb{C}[x] \end{aligned}$$

and

$$\lim_{x \to 0} (u, e^{xL(-1)}v) = (u, v).$$

4. Truncation: Pick $N \in \mathbb{Z}$ such that $V_{(n)} = 0$ for all $n \leq N$. Given $u \in V_{(r)}$, let $v \in V_{(s)}$ and $w \in V_{(t)}$. Then we have

$$\begin{aligned}(\Lambda(x)u, v \otimes w) &= (u, Y(v, x)w) \\ &= \sum_{k \in \mathbb{Z}} (u, v_k w) x^{-k-1}.\end{aligned}$$

For $(u, v_k w)$ to be nonzero, we must have $\text{wt } u = \text{wt } v + \text{wt } w - k - 1$, i.e. $r = s + t - k - 1$; but $s, t > N$, and thus we must have $r > 2N - k - 1$ or $r - 2N > -k - 1$. Hence, $(\Lambda(x)u, v \otimes w) \in \mathbb{C}[[x^{-1}]]x^{r-2N-1}$ for any $s, t \in \mathbb{Z}$.

5. Jacobi identity: Given $u \in V$, then for all $v_1, v_2, v_3 \in V$

$$\begin{aligned}((Id_V \otimes \Lambda(x_2))\Lambda(x_1)u, v_1 \otimes v_2 \otimes v_3) &= (\Lambda(x_1)u, v_1 \otimes Y(v_2, x_2)v_3) \\ &= (u, Y(v_1, x_1)Y(v_2, x_2)v_3),\end{aligned} \tag{9.2.4}$$

$$\begin{aligned}((T \otimes Id_V)(Id_V \otimes \Lambda(x_1))\Lambda(x_2)u,&v_1 \otimes v_2 \otimes v_3) \\ &= ((Id_V \otimes \Lambda(x_1))\Lambda(x_2)u, v_2 \otimes v_1 \otimes v_3) \\ &= (\Lambda(x_2)u, v_2 \otimes Y(v_1, x_1)v_3) \\ &= (u, Y(v_2, x_2)Y(v_1, x_1)v_3),\end{aligned} \tag{9.2.5}$$

$$\begin{aligned}((\Lambda(x_0) \otimes Id_V)\Lambda(x_2)u, v_1 \otimes v_2 \otimes v_3) &= (\Lambda(x_2)u, Y(v_1, x_0)v_2 \otimes v_3) \\ &= (u, Y(Y(v_1, x_0)v_2, x_2)v_3).\end{aligned} \tag{9.2.6}$$

Equations (9.2.4), (9.2.5) and (9.2.6) make it clear that the VOA Jacobi identity (9.1.5) is equivalent to the VOC Jacobi identity (7.1.5).

6. Virasoro algebra: Given $u \in V$, for all $v \in V$

$$\begin{aligned}((\rho \otimes Id_V)\Lambda(x)u, v) &= (\Lambda(x)u, \omega \otimes v) \\ &= (u, Y(\omega, x)v) \\ &= \sum_{k \in \mathbb{Z}} (u, L(k)v)x^{-k-2} \\ &= \sum_{j \in \mathbb{Z}} (L(j)u, v)x^{j-2}.\end{aligned} \tag{9.2.7}$$

Note that in the last equality we have used Virasoro preservation, (9.2.1).

Equation (9.2.7) shows that the Virasoro algebra bracket,

$$[L(j), L(k)] = (j-k)L(j+k) + \frac{1}{12}(j^3 - j)\delta_{j,-k}d,$$

follows from the Virasoro bracket relation on VOAs.

7. Grading: Equation (9.2.7) shows that $L(0) = Res_x xY(\omega, x)$ so grading follows from VOAs.

8. $L(1)$-derivative: Given $u \in V$, then for all $v, w \in V$

$$\begin{aligned}((L(1) \otimes Id_V)\Lambda(x)u, v \otimes w) &= (\Lambda(x)u, L(-1)v \otimes w) \\ &= (u, Y(L(-1)v, x)w) \\ &= \frac{d}{dx}(u, Y(v, x)w) \\ &= \frac{d}{dx}(\Lambda(x)u, v \otimes w).\end{aligned}$$

Here the first equality uses Virasoro preservation. □

Li showed in [L] that if a simple VOA satisfies the condition $L(1)V_{(1)} = 0$ then there exists a nondegenerate, invariant bilinear form on V. Thus we are guaranteed a family of VOAs equipped with the type of form required for Theorem 9.2.1. Additionally, Heisenberg VOAs may be explicitly equipped with an appropriate bilinear form and they will be the focus of more concrete discussion in the next section.

9.3 Vertex operator algebras and coalgebras associated with Heisenberg algebras

While the construction in the last section does describe a family of VOCs, it is not extremely explicit in nature. In this section we will explicitly construct VOCs from Heisenberg algebras. We begin with the construction of the vector space for the Heisenberg VOA following [D].

Let $\mathbf{h}$ be a finite dimensional vector space (over a field $\mathbb{C}$ of characteristic 0) equipped with a symmetric, nondegenerate bilinear form $\langle \cdot, \cdot \rangle$. Since $\mathbf{h}$ may be considered as an abelian Lie algebra, let $\hat{\mathbf{h}}$ be the corresponding affine Lie algebra, i.e., let

$$\hat{\mathbf{h}} = \mathbf{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where c is nonzero, with the Lie bracket defined by

$$[\alpha \otimes t^m, \beta \otimes t^n] = \langle \alpha, \beta \rangle m \delta_{m,-n} c,$$

$$[\hat{\mathbf{h}}, c] = 0$$

for $\alpha, \beta \in \mathbf{h}$, $m, n \in \mathbb{Z}$. There is a natural $\mathbb{Z}$-grading on $\hat{\mathbf{h}}$ under which $\alpha \otimes t^m$ has weight $-m$ for all $\alpha \in \mathbf{h}$, and $m \in \mathbb{Z}$, and c has weight 0. The element $\alpha \otimes t^m$ of $\hat{\mathbf{h}}$ is usually denoted $\alpha(m)$. Three graded subalgebras are of interest:

$$\hat{\mathbf{h}}^+ = \mathbf{h} \otimes t\mathbb{C}[t],$$

$$\hat{\mathbf{h}}^- = \mathbf{h} \otimes t^{-1}\mathbb{C}[t^{-1}],$$

$$\hat{\mathbf{h}}_{\mathbb{Z}} = \hat{\mathbf{h}}^+ \oplus \hat{\mathbf{h}}^- \oplus \mathbb{C}c.$$

The subalgebra $\hat{\mathbf{h}}_{\mathbb{Z}}$ is a *Heisenberg algebra*, by which we mean that its center is one-dimensional and is equal to its commutator subalgebra. Note that $\hat{\mathbf{h}}^+$ and $\hat{\mathbf{h}}^-$ are abelian, but that $\hat{\mathbf{h}}_{\mathbb{Z}}$ is necessarily non-abelian.

We consider the induced $\hat{\mathbf{h}}_{\mathbb{Z}}$-module

$$M(1) = U(\hat{\mathbf{h}}_{\mathbb{Z}}) \otimes_{U(\hat{\mathbf{h}}^+ \oplus \mathbb{C}c)} \mathbb{C}$$

where U indicates the universal enveloping algebra and $\mathbb{C}$ is viewed as a $\mathbb{Z}$-graded $(\hat{\mathbf{h}}^+ \oplus \mathbb{C}c)$-module by

$$c \cdot 1 = 1,$$
$$\hat{\mathbf{h}}^+ \cdot 1 = 0,$$
$$\deg 1 = 0.$$

The module $M(1)$ may be generalized (cf. [L] and [FLM]). $M(1)$ is linearly isomorphic to $S(\hat{\mathbf{h}}^-)$ in a way that preserves grading. Thus we often write basis elements of $M(1)$ as

$$v = \alpha_1(-n_1)\cdots\alpha_r(-n_r)$$

for $\alpha_i \in \mathbf{h}$, $n_i \in \mathbb{Z}_+$, $i = 1, \dots, r$, and observe that v has weight $n_1 + \cdots + n_r$. (Tensor products are suppressed in this notation.) Note that the $\alpha_i(-n_i)$'s all commute so their order is irrelevant. Using the form $\langle \cdot, \cdot \rangle$ on $\mathbf{h}$, we may choose $\{\gamma_i\}_{i=1}^d$ to be an orthonormal basis and we lose no generality by considering only α_i from this set of basis elements. Thus, we will typically prove results for the set of generating elements

$$\text{gen}M = \left\{\alpha_1(-n_1)\cdots\alpha_r(-n_r) | r \in \mathbb{N},\ \alpha_j \in \{\gamma_i\}_{i=1}^d,\ n_j \in \mathbb{Z}_+,\ j = 1, \dots, r\right\}$$

and then extend linearly to all of $M(1)$.

Next we define a bilinear form on $M(1)$ which we will use throughout the rest of this section.

Lemma 9.3.1. *There is a unique bilinear form $(\cdot, \cdot)$ on $M(1)$ satisfying*

$$(\alpha(m) \cdot u, v) = (u, \alpha(-m) \cdot v), \tag{9.3.1}$$
$$(1, 1) = 1. \tag{9.3.2}$$

for all $u, v \in M(1)$, $\alpha \in \mathbf{h}$, $m \in \mathbb{Z} \setminus \{0\}$.

More precisely, given $v = \alpha_1(-n_1)\cdots\alpha_r(-n_r)$ *let*

$$p(v) = \alpha_r(n_r)\cdots\alpha_1(n_1)\alpha_1(-n_1)\cdots\alpha_r(-n_r) \in \mathbb{Z}_+ \subset M(1).$$

The unique bilinear form $(\cdot,\cdot)$ *on* $M(1)$ *satisfying (9.3.1) and (9.3.2) is defined on basis elements* $u, v \in genM$ *by*

$$(u,v) = \begin{cases} p(u) & \text{if } u = v \\ 0 & \text{otherwise} \end{cases} \tag{9.3.3}$$

Further, this form is nondegenerate, graded and symmetric.

Proof. We will construct the form in (9.3.3) from (9.3.1) and (9.3.2), thus showing the form is unique. Consider $u, v \in \text{gen}M$ such that $u \neq v$. Then there is an element $\alpha(-n) \in \hat{\mathbf{h}}$ and a positive integer t such that $\alpha(-n)^t$ is contained in u or v but not in the other. We may assume $\alpha(-n)^t$ is in u, say $u = \alpha(-n)^t\alpha_1(-n_1)\cdots\alpha_r(-n_r)$. But then

$$\begin{aligned}(u,v) &= (\alpha(-n)^t\alpha_1(-n_1)\cdots\alpha_r(-n_r), v)\\ &= (\alpha_1(-n_1)\cdots\alpha_r(-n_r), \alpha(n)^t v)\\ &= (\alpha_1(-n_1)\cdots\alpha_r(-n_r), 0)\\ &= 0.\end{aligned} \tag{9.3.4}$$

(As a byproduct, (9.3.4) shows the form is graded and symmetric.) Now we need only examine the form applied to a single basis element. Let $u = \alpha_1(-n_1)\cdots\alpha_r(-n_r)$. Then

$$\begin{aligned}(u,u) &= (\alpha_1(-n_1)\cdots\alpha_r(-n_r), \alpha_1(-n_1)\cdots\alpha_r(-n_r))\\ &= (\alpha_r(n_r)\cdots\alpha_1(n_1)\alpha_1(-n_1)\cdots\alpha_r(-n_r), 1)\\ &= (p(u), 1)\\ &= p(u),\end{aligned} \tag{9.3.5}$$

thus proving that (9.3.3) is the unique form satisfying (9.3.1) and (9.3.2). Nondegeneracy is immediate from (9.3.5). □

Given $\alpha \in \mathbf{h}$, we will need the following three series in a formal variable, x, with coefficients in $\hat{\mathbf{h}}_{\mathbb{Z}}$:

$$\alpha^+(x) = \sum_{k \in \mathbb{Z}_+} \alpha(k) x^{-k-1},$$

$$\alpha^-(x) = \sum_{k \in \mathbb{Z}_+} \alpha(-k) x^{k-1},$$

$$\alpha(x) = \alpha^-(x) + \alpha^+(x).$$

The vertex operator algebra associated to a Heisenberg algebra is described using a normal ordering procedure, indicated by open colons $\circ\atop\circ$ $\circ\atop\circ$, which reorders the enclosed expression so that all operators $\alpha_1(-m)$ are placed to the left (multiplicatively) of all operators $\alpha_2(n)$, for $\alpha_1, \alpha_2 \in \mathbf{h}$, $m, n \in \mathbb{Z}_+$. For example,

$$\begin{aligned} {\circ\atop\circ}\alpha_1(x)\alpha_2(x){\circ\atop\circ}v &= {\circ\atop\circ}(\alpha_1^-(x) + \alpha_1^+(x))\alpha_2(x){\circ\atop\circ}v \\ &= \alpha_1^-(x)\alpha_2(x)v + \alpha_2(x)\alpha_1^+(x)v. \end{aligned}$$

This normal ordering moves degree-lowering operators to the right. Notice that $\alpha^+(x)$ will only produce elements of lower weight than the basis element of $M(1)$ to which it is applied, while $\alpha^-(x)$ will only produce elements of higher weight. Therefore applying all the degree-lowering operators before all the degree-raising operators guarantees that no single weight-space has infinitely many summands in it.

We now have the notation to describe the VOA associated to a Heisenberg algebra. Define a linear map $Y(\cdot, x) : M(1) \to (\text{End } M(1))[[x, x^{-1}]]$ by

$$Y(v, x) = {\circ\atop\circ} \left(\frac{1}{(n_1 - 1)!} \left(\frac{d}{dx} \right)^{n_1 - 1} \alpha_1(x) \right) \cdots \left(\frac{1}{(n_r - 1)!} \left(\frac{d}{dx} \right)^{n_r - 1} \alpha_r(x) \right) {\circ\atop\circ}$$

for $v = \alpha_1(-n_1) \cdots \alpha_r(-n_r)$. We also define two distinguished elements of $M(1)$, $\mathbf{1} = 1$ and $\omega = \frac{1}{2} \sum_{i=1}^{d} \gamma_i(-1)^2$, where $\{\gamma_i\}_{i=1}^{d}$ is the orthonormal basis of $\mathbf{h}$ as above.

Proposition 9.3.2. *The quadruple $(M(1), Y, \mathbf{1}, \omega)$ as defined above is a vertex operator algebra.*

Our treatment here largely mirrors [D], with the above proposition being Proposition 3.1 in [D]. A proof may be found in [G] or [LL]. For our purposes, however, the nondegenerate bilinear form on $M(1)$, described in (9.3.3), is equally relevant. We will now prove an additional property of that bilinear form.

Lemma 9.3.3. *The bilinear form on $M(1)$ defined in Equation (9.3.3) is Virasoro preserving.*

Proof. First, we will explicitly calculate the $L(k)$ operators and then show that $(L(k)v, w) = (v, L(-k)w)$. Symmetry of the form allows us to only consider $k \in \mathbb{N}$.

Using the definition of the $L(k)$ operators we see that

$$\sum_{k\in\mathbb{Z}} L(k)x^{-k-2} = Y(\frac{1}{2}\sum_{i=1}^{d}\gamma_i(-1)^2, x).$$

Employing the definition of Y, for $k \in \mathbb{Z}_+$ we have

$$L(k) = \sum_{i=1}^{d}\left(\frac{1}{2}\sum_{j=1}^{k-1}\gamma_i(j)\gamma_i(k-j) + \sum_{j\in\mathbb{Z}_+}\gamma_i(-j)\gamma_i(k+j)\right)$$
$$L(-k) = \sum_{i=1}^{d}\left(\frac{1}{2}\sum_{j=1}^{k-1}\gamma_i(-j)\gamma_i(-k+j) + \sum_{j\in\mathbb{Z}_+}\gamma_i(-k-j)\gamma_i(j)\right).$$

Given $u, v \in \text{gen}M$,

$$(L(0)u, v) = \text{wt }(u)\ (u, v) = \text{wt }(v)\ (u, v) = (u, L(0)v)$$

since either $(u, v) = 0$, or $u = v$ implying wt $(u) =$ wt (v). For $k \in \mathbb{Z}_+$ we make use of (9.3.1) to observe that,

$$\begin{aligned}(L(k)u, v) &= \sum_{i=1}^{d}\left(\frac{1}{2}\sum_{j=1}^{k-1}(\gamma_i(j)\gamma_i(k-j)u, v) + \sum_{j\in\mathbb{Z}_+}(\gamma_i(-j)\gamma_i(k+j)u, v)\right)\\ &= \sum_{i=1}^{d}\left(\frac{1}{2}\sum_{j=1}^{k-1}(u, \gamma_i(-k+j)\gamma_i(-j)v) + \sum_{j\in\mathbb{Z}_+}(u, \gamma_i(-k-j)\gamma_i(j)v)\right)\\ &= (u, L(-k)v).\end{aligned}$$

$\square$

Using the Heisenberg VOA $M(1)$ along with the nondegenerate, Virasoro preserving bilinear form defined in (9.3.3), we will follow the construction of a VOC in Section 9.2. First, we define a linear map $c : V \to \mathbb{C}$ by

$$c(1) = 1$$

$$c(\alpha_1(-n_1) \cdots \alpha_r(-n_r)) = 0$$

where $r \geq 1$ and $\alpha_1(-n_1) \cdots \alpha_r(-n_r) \in \text{gen}M$. It is clear that $c(v) = (v, \mathbf{1})$ for all $v \in M(1)$. Next, we define a linear map $\rho : V \to \mathbb{C}$ by

$$\rho(\gamma_i(-1)^2) = 1$$

for each basis element γ_i of $\mathbf{h}$ and $\rho(v) = 0$ for v any other basis element of $M(1)$. Again it is clear that $\rho(v) = (v, \omega)$ for all $v \in M(1)$. Finally, we need to define a linear map $\varLambda(x) : V \to (V \otimes V)[[x, x^{-1}]]$ such that

$$(\varLambda(x)u, v \otimes w) = (u, Y(v, x)w).$$

For notational simplicity, given $\alpha \in \mathbf{h}$, $n \in \mathbb{Z}_+$ and x a formal variable, let

$$\alpha^-(n, x) = \sum_{k \in \mathbb{Z}_+} \binom{k-1}{n-1} \alpha(-k) x^{k-n}$$
$$\alpha^+(n, x) = \sum_{k \in \mathbb{Z}_+} \binom{-k-1}{n-1} \alpha(k) x^{-k-n}$$
$$\alpha(n, x) = \alpha^-(n, x) + \alpha^+(n, x)$$

so that

$$Y(\alpha_1(-n_1) \cdots \alpha_r(-n_r), x) = {}^\circ_\circ \alpha_1(n_1, x) \cdots \alpha_r(n_r, x) {}^\circ_\circ.$$

It is also notationally useful to define

$$\alpha_{\bullet}^{-}(n,x) = \sum_{k\in\mathbb{Z}_+} \binom{-k-1}{n-1} \alpha(-k)x^{-k-n}$$

$$\alpha_{\bullet}^{+}(n,x) = \sum_{k\in\mathbb{Z}_+} \binom{k-1}{n-1} \alpha(k)x^{k-n}$$

$$\alpha_{\bullet}(n,x) = \alpha_{\bullet}^{-}(n,x) + \alpha_{\bullet}^{+}(n,x)$$

so that, by (9.3.1),

$$(v_1, \alpha^{-}(n,x)v_2) = (\alpha_{\bullet}^{+}(n,x)v_1, v_2), \tag{9.3.6}$$

$$(v_1, \alpha^{+}(n,x)v_2) = (\alpha_{\bullet}^{-}(n,x)v_1, v_2), \tag{9.3.7}$$

$$(v_1, \alpha(n,x)v_2) = (\alpha_{\bullet}(n,x)v_1, v_2). \tag{9.3.8}$$

Proposition 9.3.4. *Let v denote the basis element $\beta_1(-m_1)\cdots\beta_s(-m_s)$ and define $\Lambda(x) : V \to (V \otimes V)[[x, x^{-1}]]$ as*

$$\Lambda(x)u = \sum_{v\in genM} \frac{1}{p(v)} v \otimes {}^{\circ}_{\circ}\beta_{1\bullet}(m_1,x)\cdots\beta_{s\bullet}(m_s,x){}^{\circ}_{\circ}u.$$

For all $u, v, w \in M(1)$, $(\Lambda(x)u, v\otimes w) = (u, Y(v,x)w)$.

Proof. First, note that our definition of Λ is equivalent to

$$(\Lambda(x)u, v\otimes w) = (\frac{1}{p(v)} v \otimes {}^{\circ}_{\circ}\beta_{1\bullet}(m_1,x)\cdots\beta_{s\bullet}(m_s,x){}^{\circ}_{\circ}u, v\otimes w)$$

for all $u = \alpha_1(-n_1)\cdots\alpha_r(-n_r)$, $v = \beta_1(-m_1)\cdots\beta_s(-m_s)$, $w = \mu_1(-\ell_1)\cdots\mu_t(-\ell_t)$. Given these basis elements we use induction on s to show that

$$({}^{\circ}_{\circ}\beta_{1\bullet}(m_1,x)\cdots\beta_{s\bullet}(m_s,x){}^{\circ}_{\circ}u, w) = (u, {}^{\circ}_{\circ}\beta_1(m_1,x)\cdots\beta_s(m_s,x){}^{\circ}_{\circ}w). \tag{9.3.9}$$

For $s = 0$, this is trivial. If we assume that (9.3.9) is true for $s-1$ and appeal to (9.3.6) and (9.3.7), we see that

$$({}^{\circ}_{\circ}\beta_{1\bullet}(m_1,x)\cdots\beta_{s\bullet}(m_s,x){}^{\circ}_{\circ}u, w)$$

$$= (\beta^{-}_{s.}(m_s,x)\ {}^{\circ}_{\circ}\beta_{1.}(m_1,x)\cdots\beta_{s-1.}(m_{s-1},x)\ {}^{\circ}_{\circ}u, w)$$
$$+ (\ {}^{\circ}_{\circ}\beta_{1.}(m_1,x)\cdots\beta_{s-1.}(m_{s-1},x)\ {}^{\circ}_{\circ}\beta^{+}_{s.}(m_s,x)u, w)$$

$$= (\ {}^{\circ}_{\circ}\beta_{1.}(m_1,x)\cdots\beta_{s-1.}(m_{s-1},x)\ {}^{\circ}_{\circ}u, \beta^{+}_{s}(m_s,x)w)$$
$$+ (\beta^{+}_{s.}(m_s,x)u,\ {}^{\circ}_{\circ}\beta_{1}(m_1,x)\cdots\beta_{s-1}(m_{s-1},x)\ {}^{\circ}_{\circ}w)$$

$$= (u,\ {}^{\circ}_{\circ}\beta_{1}(m_1,x)\cdots\beta_{s-1}(m_{s-1},x){}^{\circ}_{\circ}\ \beta^{+}_{s}(m_s,x)w)$$
$$+ (u, \beta^{-}_{s}(m_s,x)\ {}^{\circ}_{\circ}\beta_{1}(m_1,x)\cdots\beta_{s-1}(m_{s-1},x)\ {}^{\circ}_{\circ}w)$$

$$= (u,\ {}^{\circ}_{\circ}\beta_{1}(m_1,x)\cdots\beta_{s}(m_s,x)\ {}^{\circ}_{\circ}w).$$

Finally, using Equation (9.3.9) we see that

$$\begin{aligned}
(\Lambda(x)u, v\otimes w) &= (\frac{1}{p(v)}v\otimes\ {}^{\circ}_{\circ}\beta_{1.}(m_1,x)\cdots\beta_{s.}(m_s,x)\ {}^{\circ}_{\circ}u, v\otimes w)\\
&= \frac{(v,v)}{p(v)}(\ {}^{\circ}_{\circ}\beta_{1.}(m_1,x)\cdots\beta_{s.}(m_s,x)\ {}^{\circ}_{\circ}u, w)\\
&= (u,\ {}^{\circ}_{\circ}\beta_{1}(m_1,x)\cdots\beta_{s}(m_s,x)\ {}^{\circ}_{\circ}w)\\
&= (u, Y(v,x)w).
\end{aligned}$$

□

Theorem 9.2.1 proves that the quadruple $(M(1), \Lambda, c, \rho)$ associated to the Heisenberg algebra $\hat{\mathbf{h}}_{\mathbb{Z}}$ is a vertex operator coalgebra.

BIBLIOGRAPHY

[A] Lars Ahlfors, *Conformal invariants: Topics in Geometric Function Theory*, McGraw-Hill, New York, 1973.

[B] Katrina Barron *The Notion of* $N = 1$ *Supergeometric Vertex Operator Superalgebra and the Isomorphism Theorem*, Communications in Contemporary Mathematics, Vol. 5, No. 4 (2003), 481-567.

[BHL] Katrina Barron, Yi-Zhi Huang, James Lepowsky, *Factorization of formal exponentials and uniformization*, Journal of Algebra **228** (2000), 551-579.

[BPZ] Alexander Belavin, A. E. Polyakov, Alexander Zamolodchikov, Infinite conformal symmetries in two-dimensional quantum field theory, *Nuclear Physics*, **B241** (1984), 333-380.

[Bo] Richard Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*, Proceedings of the National Academy of Science USA **83** (1986), 3068-3071.

[D] Chongying Dong, *Introduction to vertex operator algebras I*, Surikaisekikenkyusho Kokyuroku **904** (1995), 1-25.

[FK] Hershel M. Farkas and Irwin Kra, *Riemann Surfaces*, Second addition, Graduate Texts in Mathematics, 71, Springer-Verlag, New York, 1992.

[FHL] Igor B. Frenkel, Yi-Zhi Huang, James Lepowsky, *On Axiomatic Approaches to Vertex Operator Algebras and Modules*, Memoirs of the AMS, Number 494, Providence, Rhode Island, July 1993.

[FS] Daniel Friedan, Stephen Shenker, The analytic geometry of two-dimensional conformal field theory, *Nuclear Physics* **B281** (1987), 509-545.

[G] Hong Guo, On abelian intertwining algebras and modules, Ph. D. thesis, Rutgers University, 1994.

[H1] Yi-Zhi Huang, On the geometric interpretation of vertex operator algebras, Ph.D. thesis, Rutgers University, 1990.

[H2] Yi-Zhi Huang *Two-Dimensional Conformal Geometry and Vertex Operator Algebras*, Birkhäuser, Boston, 1997.

[H3] Yi-Zhi Huang, *A theory of tensor products for module categories for a vetex algebra, IV*, Journal of Pure and Applied Algebra **100** (1995), 173-216.

[HL1] Yi-Zhi Huang, James Lepowsky, *Operadic formulation of the notion of vertex operator algebra*, Contemporary Mathematics **175** (1994), 131-148.

[HL2] Yi-Zhi Huang, James Lepowsky, *A theory of tensor products for module categories for a vetex algebra, I*, Selecta Mathematica (New Series) **1** (1995), 699-756.

[HL3] Yi-Zhi Huang, James Lepowsky, *A theory of tensor products for module categories for a vetex algebra, II*, Selecta Mathematica (New Series) **1** (1995), 757-786.

[HL4] Yi-Zhi Huang, James Lepowsky, *A theory of tensor products for module categories for a vetex algebra, III*, Journal of Pure and Applied Algebra **100** (1995), 141-171.

[HL5] Yi-Zhi Huang, James Lepowsky, *Tensor products of modules for a vertex operator algebra and vertex tensor categories*, in: Jean-Lue Brylinski, Ranee Brylinski, Victor Guillemin, Victor Kac, eds., *Lie Theory and Geometry: In Honor of Bertram Kostant*, Progress in Mathematics, Vol. 123 (Birkhäuser, Boston, 1994) 349-383.

[Hub] Keith Hubbard, *Vertex coalgebras, modules, commutativity and associativity*, in preparation.

[FLM] Igor Frenkel, James Lepowsky, Arne Meurman, *Vertex Operator Algebras and the Monster*, Academic Press, Inc., San Diego, 1988.

[L] Haisheng Li, *Symmetric invariant bilinear forms on vertex operator algebras*, Journal of Pure and Applied Algebra **96** (1994), 279-297.

[LL] James Lepowsky, Haisheng Li, *Introduction to Vertex Operator Algebras and Their Representations*, Birkhäuser, Boston, 2004

[LW] James Lepowsky, Robert L. Wilson, *Construction of the affine Lie algebra* $A_1^{(1)}$, Communications in Mathematical Physics **62** (1978), 43-53.

[M] J. Peter May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics, Number 271, Springer-Verlag, 1972.

[MSS] Martin Markl, Steve Shnider, Jim Stasheff, *Operads in Algebra, Topology and Physics*, Mathematical Surverys and Monographs, Volume 96, American Mathematical Society, 2002.

[S] Graeme Segal, The definition of conformal field theory, preprint, 1988.

[St1] James D. Stasheff, Homotopy associativity of H-spaces, I, *Transactions of the American Mathematical Society* **108** (1963), 275-292.

[St2] James D. Stasheff, Homotopy associativity of H-spaces, II, *Transactions of the American Mathematical Society* **108** (1963), 293-312.

[V] Cumrun Vafa, Conformal theories and punctured surfaces, *Phys. Lett.* **B199** (1987), 195-202.